

Spatial Multiplexing in Random Wireless Networks

Kostas Stamatiou, John G. Proakis, and James R. Zeidler

Abstract—We consider a network of transmitters, each with a receiver at a fixed distance, and locations drawn independently according to a homogeneous Poisson Point Process (PPP). The transmitters and the receivers are equipped with multiple antennas. Under a channel model that includes Rayleigh fading and path-loss, and an outage model for packet successes, we examine the performance of various spatial multiplexing techniques, namely zero-forcing (ZF), ZF with successive interference cancellation (ZF-SIC or VBLAST) and DBLAST. In each case, we determine the number of streams that maximizes the transmission capacity, defined as the maximum network throughput per unit area such that a constraint on the outage probability is satisfied. Numerical results showcase the benefit of DBLAST over ZF and VBLAST in terms of the transmission capacity. In all cases, the transmission capacity scales linearly in the number of antennas.

Index Terms—Poisson point process, spatial multiplexing, MIMO, outage probability, transmission capacity

I. INTRODUCTION

THE study of random wireless networks has recently gathered a lot of momentum in the research community, e.g., see [1]–[4]. The main motivation behind this work is the use of tools from stochastic geometry in order to derive analytical results on how different physical, medium-access-control and network layer parameters affect the network performance. A central assumption is that the network consists of a Poisson Point Process (PPP) of transmitters, and each transmitter (TX) has a corresponding receiver (RX) at a given distance. The justification for the widespread use of this model is that it allows the analytical study of an ensemble of network topologies and captures the randomness of the node locations typical in networks without infrastructure such as ad hoc and sensor networks. The metric that quantifies the network performance is the *transmission capacity*, defined as the maximum spatial density of TX-RX links, multiplied by their rate, such that a certain constraint on the packet success probability is satisfied [2], [5]. Assuming that the channel - consisting of fading and interference - is constant during a packet slot an outage model may be employed for packet successes, i.e., a packet is successfully received if the mutual information of the channel realization is greater than the desired information rate. This translates to a requirement that the signal-to-interference-and-noise-ratio (SINR) is larger than a predetermined threshold.

In the context described above, and provided that the TX and the RX are equipped with multiple antennas, the

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objective of this paper is to shed light on the performance of certain spatial multiplexing techniques which require channel knowledge at the RX side¹; zero-forcing (ZF), ZF with successive interference cancellation (ZF-SIC, also known as VBLAST [6], [7]) and DBLAST [7], [8]. The performance of these techniques is well understood for the fading and additive noise channel [7], but not so in a network environment where the interfering nodes are randomly placed.

A. Related work

The outage probability and transmission capacity for different spatial diversity techniques and single-antenna transmission was evaluated in [9]. One of the main results of this work was that, in the small outage probability regime, for maximal ratio combining (MRC) the transmission capacity scales as $N^{2/b}$, where N is the number of RX antennas and b is the path-loss exponent. The authors in [10] considered various multiple-input multiple-output (MIMO) techniques, including spatial multiplexing, as a component of a physical layer that employs frequency hopping and coding in combating interference. They arrived at similar scaling laws to [9] regarding the network throughput and the expected progress, albeit from a different analytical path. Multiple-antenna transmission with perfect channel knowledge at the transmitter was studied in [11] and the optimal number of spatial modes, in terms of maximizing the transmission capacity for a given density, was illustrated. More recently, *multi-user* techniques such as interference cancellation and space-division multiple-access have been considered in [12]–[14]. Specifically, in [13], it was shown that optimally selecting the number of cancelled nearby interferers results in a linear scaling of the transmission capacity with N , under single-antenna transmission.

B. Contributions

We first consider single-antenna (or single-stream) transmission and revisit the performance analysis of MRC in a Poisson field of interferers, deriving a compact analytical expression for the outage/success² probability. It is shown that N RX antennas provide an approximate gain of $N^{-2/b}$ in terms of spatial contention, i.e., the rate of increase of the outage probability as a function of the transmitter density, when the latter is zero. This result provides an alternative interpretation to the scaling law derived in [9].

¹This assumption is made for the sake of simplicity of the communication protocol, as feedback to the TX is not required.

²The terms “outage” and “success” probability, since complementary, are used interchangeably throughout the paper.

We then turn our attention to multiple-antenna (or multiple-stream³) transmission and, employing our findings for the single-stream scenario, derive exact expressions and approximations to the outage probability for ZF, VBLAST and DBLAST. The optimal number of streams such that the transmission capacity of the network is maximized is determined for each of these techniques in the small outage probability regime. The trade-off lies in the fact that, introducing more streams can potentially boost the information rate of each link, but also increases the interference level in the network. For DBLAST specifically, it is shown that, for $b \geq 4$, it is optimal to use all transmit antennas, while, for $b < 4$, the number of streams must be judiciously chosen such that the optimal trade-off is achieved. Numerical results indicate that the benefit of DBLAST over ZF and VBLAST is significant in terms of the transmission capacity. For all spatial multiplexing techniques, provided that the number of streams is optimally chosen, the transmission capacity scales linearly in the number of antennas.

C. Paper organization and notation

The remainder of the paper is organized as follows. In Section II we describe in detail our system model. Section III is devoted to the analysis of the single-stream scenario and Section IV covers the extensions to the multiple-stream case. Our numerical results are outlined in Section V and Section VI concludes the paper.

We note the following regarding the notation: a zero-mean complex Gaussian random vector \mathbf{x} , with covariance matrix $\mathbf{Q} = E[\mathbf{x}\mathbf{x}^H]$ is denoted as $\mathbf{x} \sim \mathcal{CN}(0, \mathbf{Q})$; the central chi-square distribution with parameter $1/2$ and $2l$, $l \in \mathcal{Z}^+$, degrees of freedom is denoted as χ_{2l}^2 ; the $l \times l$ identity and zero matrices are denoted as \mathbf{I}_l , \mathbf{O}_l , respectively; “ \propto ” stands for “proportional to”, “ \simeq ” stands for asymptotic equality and “ \approx ” denotes an approximate equality.

II. SYSTEM MODEL

The network consists of an infinite number of TXs, each with a corresponding RX at distance R , and locations $\{x_i\}$ that are drawn independently according to a homogeneous PPP $\Pi = \{x_i\}$ of density λ . Time is slotted and transmissions take place concurrently and in a synchronized manner during each slot. Due to the stationarity of the homogeneous PPP, the performance of any TX-RX link, i.e., “typical” link, may be studied. The network model, within a disc of finite radius around the typical RX, is depicted in Fig. 1.

The channel between each TX-RX pair consists of constant flat Rayleigh fading and path-loss according to the law r^{-b} , with $b > 2$ (this requirement ensures that the interference power is finite [15]). Additive noise is disregarded, hence interference from concurrent transmissions is the only cause of errors in communication⁴. The power from each antenna is the

³In this paper, the number of packet streams is equal to the number of active TX antennas. Each stream may be transmitted on the same antenna, such as in ZF or VBLAST, or across different antennas, as in DBLAST.

⁴We select to study an interference-limited scenario in order to focus on the effect of cochannel interference on the performance of the employed physical-layer techniques. The analysis can be generalized to include thermal noise.

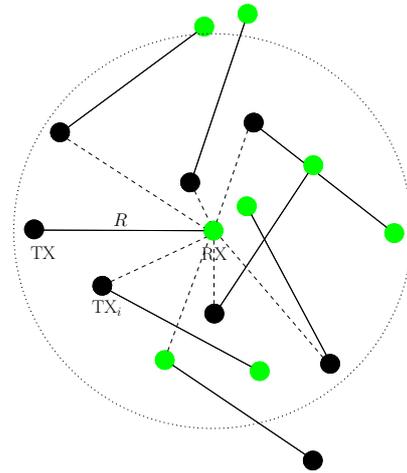


Fig. 1. Network model. The black circles denote the transmitters and the green circles the corresponding receivers at distance R . Solid/dashed lines denote useful/interfering signals.

same across all transmitters and, due to the absence of noise, may obtain an arbitrary value, e.g., unity. Generally, there is a different number of antennas at the TX and the RX; however, for convenience, we assume that N antennas are available at both the TX and the RX⁵.

Suppose that M antennas are employed for transmission, with $M \leq N$. The received vector at the typical RX can be written as

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w}, \quad (1)$$

where \mathbf{H} is the $N \times M$ channel matrix between TX and RX, with i.i.d. elements $[\mathbf{H}]_{nm} \sim \mathcal{CN}(0, 1)$; $\mathbf{x} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_M)$ is the $M \times 1$ symbol vector transmitted by TX; and \mathbf{w} is the interference term, modeled as $\mathbf{w} \sim \mathcal{CN}(\mathbf{0}, z\mathbf{I}_N)$, where

$$z = MR^{\frac{b}{2}} \sum_{x_i \in \Pi \setminus \{x_0\}} R_i^{-b} \quad (2)$$

is the total interference power over a given slot, per RX antenna; x_0 denotes the location of the typical TX and R_i is the distance of the interfering TX at location x_i from the typical RX⁶. It is known that z is an α -stable random variable with stability exponent $\alpha = 2/b$ [1], [4]. Its moment generating function (mgf) is given by

$$\Phi_z(s) = E[e^{-sz}] = e^{-cs^\alpha}, \quad s > 0, \quad (3)$$

where the parameter c is defined as $c \triangleq \lambda\pi R^2\Gamma(1-\alpha)M^\alpha$ and $\Gamma(x)$, $x > 0$ denotes the gamma function.

III. SINGLE-ANTENNA TRANSMISSION ($M = 1$)

Consider the transmission of a single stream, i.e., $M = 1$ and $c = \lambda\pi R^2\Gamma(1-\alpha)$. Defining the desirable information

⁵This assumption is reasonable in an ad hoc network, where a node can be a TX or a RX at different times.

⁶Note that, taking into account the fading from an interferer to a typical RX, the interference is generally correlated across the RX antennas. Assuming the interference is uncorrelated is a worst-case scenario, which simplifies the analysis.

rate as $\mathcal{R} = \log(1 + \theta)$, where θ is an appropriate signal-to-interference-ratio (SIR) threshold, the success probability corresponding to (1) is given by [7]

$$\begin{aligned} P_s &= \text{P} \left(\log \left(1 + \frac{a}{z} \right) > \mathcal{R} \right) \\ &= \text{P} \left(\frac{a}{z} > \theta \right), \end{aligned} \quad (4)$$

where $a = \|\mathbf{H}\|^2$ is chi-square distributed with $2N$ degrees of freedom, i.e., $a \sim \chi_{2N}^2$. The respective outage probability is $P_o = 1 - P_s$.

A. Evaluation of P_s

The evaluation of P_s requires the knowledge of the statistics of the SIR $\gamma = a/z$. In the following theorem, the complementary cumulative distribution (ccdf) of γ is derived.

Theorem 1 *Let $\gamma = a/z$, where $a \sim \chi_{2N}^2$ and z is an α -stable random variable with mgf given by (3). The ccdf of γ , $\bar{F}_\gamma(x)$, is given by*

$$\bar{F}_\gamma(x) = e^{-cx^\alpha} + e^{-cx^\alpha} \sum_{k=1}^{N-1} \frac{(cx^\alpha)^k}{k!} \sum_{n=k}^{N-1} \frac{|\beta_k^n|}{n!}, \quad x > 0, \quad (5)$$

where

$$\beta_k^n = \sum_{m=1}^k (-1)^m \binom{k}{m} (\alpha m)_n, \quad k = 1, \dots, n \quad (6)$$

and $(\alpha m)_n \triangleq \alpha m \dots (\alpha m - n + 1)$ is the falling sequential product.

Proof: By the definition of $\bar{F}_\gamma(x)$, we have that

$$\bar{F}_\gamma(x) = \text{P}(a > xz) = \int_0^{+\infty} \bar{F}_a(xy) f_z(y) dy, \quad (7)$$

where $\bar{F}_a(t)$ is the ccdf of a , given by

$$\bar{F}_a(t) = e^{-t} \sum_{n=0}^{N-1} \frac{t^n}{n!} = \frac{\Gamma(N, t)}{(N-1)!}, \quad t > 0. \quad (8)$$

Substituting (8) in (7), we obtain

$$\bar{F}_\gamma(x) = \Phi_z(x) + \sum_{n=1}^{N-1} \frac{x^n}{n!} \int_0^{+\infty} y^n f_z(y) e^{-xy} dy.$$

From the Laplace transform property

$$f_z(y) y^n \xrightarrow{\mathcal{L}} (-1)^n \frac{d^n \Phi_z(s)}{ds^n}, \quad (9)$$

it follows that

$$\bar{F}_\gamma(x) = \Phi_z(x) + \sum_{n=1}^{N-1} \frac{x^n}{n!} (-1)^n \frac{d^n \Phi_z(x)}{dx^n}. \quad (10)$$

Using identity 0.430.1, p.24, [16] for the n^{th} derivative of a composite function, after some algebra, we obtain

$$\frac{d^n \Phi_z(x)}{dx^n} = x^{-n} e^{-cx^\alpha} \sum_{k=1}^n \frac{\beta_k^n}{k!} (cx^\alpha)^k, \quad (11)$$

where β_k^n is defined in (6). Substituting (11) in (10) and regrouping terms results in

$$\begin{aligned} \bar{F}_\gamma(x) &= e^{-cx^\alpha} + e^{-cx^\alpha} \sum_{n=1}^{N-1} \frac{1}{n!} \sum_{k=1}^n \frac{(-1)^n \beta_k^n}{k!} (cx^\alpha)^k \\ &= e^{-cx^\alpha} + e^{-cx^\alpha} \sum_{k=1}^{N-1} \frac{(cx^\alpha)^k}{k!} \sum_{n=k}^{N-1} \frac{(-1)^n \beta_k^n}{n!}. \end{aligned} \quad (12)$$

In order to arrive at (5), we now need to show that $(-1)^n \beta_k^n \geq 0$. Once again, using the identity for the n^{th} derivative of a composite function, β_k^n can be written as the following derivative evaluated at $x = 1$.

$$\beta_k^n = \left. \frac{d^n (1 - x^\alpha)^k}{dx^n} \right|_{x=1}. \quad (13)$$

From (13), the following iterative relation can be proved for $n \geq 2$

$$\beta_k^n = \sum_{m_1=1}^n \binom{n}{m_1} \beta_1^{m_1} \beta_{k-1}^{n-m_1}. \quad (14)$$

By successive application of (14), we obtain

$$\begin{aligned} \frac{(-1)^n \beta_k^n}{n!} &= \\ &= \sum_{m_1=1}^n \sum_{m_2=1}^{n-m_1} \dots \sum_{m_{k-1}=1}^{n-m_k-2-\dots-m_1} (-1)^{m_1} \beta_1^{m_1} (-1)^{m_2} \beta_1^{m_2} \dots (-1)^{m_k} \beta_1^{m_k}, \end{aligned} \quad (15)$$

where $m_k = n - m_{k-1} - \dots - m_1$. However, $(-1)^n \beta_1^n \geq 0$, since, by (6), $(-1)^n \beta_1^n = (-1)^{n+1} \alpha(\alpha-1) \dots (\alpha-n+1)$ and $\alpha = 2/b < 1$. Therefore, $(-1)^n \beta_k^n \geq 0$ for $k = 1, \dots, n$. ■

By the definition of P_s in (4), we have that $P_s = \bar{F}_\gamma(\theta)$ or

$$P_s = e^{-c\theta^\alpha} + e^{-c\theta^\alpha} \sum_{k=1}^{N-1} \frac{(c\theta^\alpha)^k}{k!} \sum_{n=k}^{N-1} \frac{|\beta_k^n|}{n!}. \quad (16)$$

We can see that P_s is a product of the term $e^{-c\theta^\alpha}$ (the success probability for $N = 1$) and a polynomial in $c\theta^\alpha$ of degree $N - 1$ and non-negative coefficients. Clearly, increasing the number of antennas N , increases the success probability as more positive terms are added to the polynomial.

In order to obtain more insight into the effect of $N > 1$ on the success probability, we evaluate the *spatial contention parameter*

$$\eta = - \left. \frac{\partial P_s}{\partial \lambda} \right|_{\lambda=0}, \quad (17)$$

defined in [17] for single-antenna networks as the slope of the outage probability as a function of the density λ , at $\lambda = 0$. By its definition, the larger η is, the sharper the increase of the outage probability as λ increases. We have the following proposition.

Proposition 1 *In a network with single-antenna transmission ($M = 1$), the spatial contention parameter η is*

$$\eta = \frac{c\theta^\alpha}{\lambda} \frac{\Gamma(N - \alpha)}{\Gamma(N)\Gamma(1 - \alpha)}. \quad (18)$$

Proof: From the definition of η and (16), we have

$$\begin{aligned} \eta &= \frac{c\theta^\alpha}{\lambda} - \frac{c\theta^\alpha}{\lambda} \sum_{n=1}^{N-1} \frac{|\beta_1^n|}{n!} \\ &= \frac{c\theta^\alpha}{\lambda} + \frac{c\theta^\alpha}{\lambda} \sum_{n=1}^{N-1} \frac{(-1)^n (\alpha)_n}{n!} \\ &= \frac{c\theta^\alpha}{\lambda} \sum_{n=0}^{N-1} \frac{(-1)^n (\alpha)_n}{n!} \quad | (\alpha)_0 \triangleq 1 \\ &= \frac{c\theta^\alpha}{\lambda} \frac{(-1)^{N-1}}{(N-1)!} \sum_{n=0}^{N-1} \binom{N-1}{n} (N-1-n)! (-1)^{N-1-n} (\alpha)_n \\ &= \frac{c\theta^\alpha}{\lambda} \frac{(-1)^{N-1}}{(N-1)!} \sum_{n=0}^{N-1} \binom{N-1}{n} (-1)^{N-1-n} (\alpha)_n \\ &= \frac{c\theta^\alpha}{\lambda} \frac{(-1)^{N-1}}{(N-1)!} (\alpha - 1)_{N-1} \end{aligned} \quad (19)$$

$$= \frac{c\theta^\alpha}{\lambda} \frac{\Gamma(N - \alpha)}{\Gamma(N)\Gamma(1 - \alpha)} \quad (20)$$

where (19) stems from the binomial identity for falling sequential products and (20) is the result of the successive application of the gamma function property $\Gamma(x + 1) = x\Gamma(x)$. ■

For increasing values of N , Stirling's approximation yields

$$\frac{\Gamma(N - \alpha)}{\Gamma(N)} \simeq N^{-\alpha} \left(1 - \frac{\alpha}{N}\right)^{N - \alpha - \frac{1}{2}} e^\alpha.$$

However, it is easy to verify that $\lim_{N \rightarrow \infty} \left(1 - \frac{\alpha}{N}\right)^{N - \alpha - \frac{1}{2}} = e^{-\alpha}$, so

$$\frac{\Gamma(N - \alpha)}{\Gamma(N)} \simeq N^{-\alpha}. \quad (21)$$

As a result, for increasing N , $\eta \simeq \pi R^2 \theta^\alpha N^{-\alpha}$. In other words, N provides an approximate gain of $N^{-\alpha}$ in terms of spatial contention, or, equivalently, the antenna array at the RX effectively decreases the SIR threshold θ by a factor N . Note that (21) is quite accurate for relatively small values of N , e.g., for $N = 5$, the error is of the order of 10% for $b = 4$, and 5% for $b = 6$.

To conclude the analysis of the single-stream scenario, the following proposition provides an upper bound on P_s which is tight as $\lambda \rightarrow 0$.

Proposition 2 *For single-antenna transmission, P_s is upper-bounded as*

$$P_s \leq \exp(-\eta\lambda) \triangleq P_u(\lambda, N) \quad (22)$$

where the equality holds for $N = 1$. Furthermore, for $\lambda \rightarrow 0$, $P_s \approx P_u(\lambda, N)$.

Proof: For $N = 1$, $P_s = e^{-c\theta^\alpha}$, so (22) holds as an equality. For $N > 1$, we take the Taylor series expansion of

$P_u(\lambda, N)$ over λ and compare individual terms with (16). The reader may verify that it suffices to prove that

$$A_k \triangleq \sum_{n=k}^{N-1} \frac{(-1)^n \beta_k^n}{n!} \leq \left(\sum_{n=1}^{N-1} \frac{(-1)^n \beta_1^n}{n!} \right)^k \triangleq B_k, \quad (23)$$

with $k = 1, \dots, N - 1$. If $k = 1$, (23) holds as an equality. For $k > 1$, it holds that

$$B_k = \sum_{n_1=1}^{N-1} \dots \sum_{n_k=1}^{N-1} \delta_1^{n_1} \dots \delta_1^{n_k} \quad (24)$$

where, for convenience, we have defined $\delta_1^n = \frac{(-1)^n \beta_1^n}{n!}$. Moreover, by (15), we have that

$$\frac{(-1)^n \beta_k^n}{n!} = \sum_{m_1=1}^n \sum_{m_2=1}^{n-m_1} \dots \sum_{m_{k-1}=1}^{n-m_1-\dots-m_{k-2}} \delta_1^{m_1} \delta_1^{m_2} \dots \delta_1^{m_k} \quad (25)$$

where $m_k = n - m_{k-1} - \dots - m_1$. Substituting (25) and (24) in (23), we can see that (23) is a true statement. This is due to the fact that the summation that gives A_k is over a subset of the terms that are summed to give B_k .

Finally, by the definition of η , for $\lambda \rightarrow 0$, $P_s \simeq 1 - \eta\lambda \simeq P_u(\lambda, N)$. ■

As a result of Proposition 2, (22) can be used as an approximation to P_s in the small outage probability regime.

B. Transmission capacity ($M = 1$)

We utilize the results of the previous subsection in evaluating the transmission capacity of the network, defined as the maximum network throughput per unit area, such that a constraint $P_s = 1 - \epsilon$ is satisfied [2], [3], i.e.,

$$\text{TC}_\epsilon = \lambda_\epsilon (1 - \epsilon) \mathcal{R}, \quad (26)$$

where the *maximum contention density* λ_ϵ is determined by the constraint $P_s = 1 - \epsilon$. In the small outage probability regime, e.g., typically, $\epsilon \leq 0.1$, we can invoke Proposition 2 to derive the following approximation to λ_ϵ

$$\begin{aligned} P_u(\lambda_\epsilon, N) &\approx 1 - \epsilon \\ \exp\left(-\frac{\Gamma(N - \alpha)}{\Gamma(N)} \lambda_\epsilon \pi R^2 \theta^\alpha\right) &\approx 1 - \epsilon \\ \lambda_\epsilon &\approx -\frac{\log(1 - \epsilon)}{\pi R^2 \theta^\alpha} \frac{\Gamma(N)}{\Gamma(N - \alpha)}. \end{aligned} \quad (27)$$

From (26) and (27), an approximation to the transmission capacity is thus

$$\text{TC}_\epsilon \approx \log(1 + \theta) \frac{(\epsilon - 1) \log(1 - \epsilon)}{\pi R^2 \theta^\alpha} \cdot \frac{\Gamma(N)}{\Gamma(N - \alpha)}. \quad (28)$$

As seen by (28) and (21), the transmission capacity of a single-stream system scales as $\Theta(N^\alpha)$ in the number of RX antennas N .

IV. MULTIPLE-ANTENNA TRANSMISSION ($M > 1$)

A. ZF

We now turn our attention to the case $M > 1$ - hence $c = \lambda\pi R^2\Gamma(1-\alpha)M^\alpha$. Assume that each packet is transmitted over the same antenna during a slot with a rate $\mathcal{R} = \log(1+\theta)$. If ZF is employed at the RX, the success probability for each stream, P_s^{zf} , is also given by (4), with the difference that a is now chi-square distributed with $2(N - M + 1)$ degrees of freedom, as $M - 1$ degrees of freedom are sacrificed in order to cancel out inter-stream interference [7]. As a result, invoking (16),

$$P_s^{\text{zf}} = e^{-c\theta^\alpha} + e^{-c\theta^\alpha} \sum_{k=1}^{N-M} \frac{(c\theta^\alpha)^k}{k!} \sum_{n=k}^{N-M} \frac{|\beta_k^n|}{n!}. \quad (29)$$

From Proposition 2, we also have that $P_s^{\text{zf}} \leq P_u(\lambda, N - M + 1)$.

The transmission capacity is now defined as

$$\text{TC}_\epsilon^{\text{zf}} = \lambda_\epsilon^{\text{zf}}(1 - \epsilon)M\mathcal{R}, \quad (30)$$

where the maximum contention density $\lambda_\epsilon^{\text{zf}}$ is determined by the constraint $P_u(\lambda_\epsilon^{\text{zf}}, N - M + 1) \approx 1 - \epsilon$ for small ϵ , or

$$\lambda_\epsilon^{\text{zf}} \approx -\frac{\log(1 - \epsilon)}{\pi R^2 \theta^\alpha} \frac{\Gamma(N - M + 1)}{\Gamma(N - M + 1 - \alpha)M^\alpha}. \quad (31)$$

The transmission capacity is thus given by

$$\text{TC}_\epsilon^{\text{zf}} \approx \log(1 + \theta) \frac{(\epsilon - 1) \log(1 - \epsilon)}{\pi R^2 \theta^\alpha} \cdot \frac{\Gamma(N - M + 1)M^{1-\alpha}}{\Gamma(N - M + 1 - \alpha)}. \quad (32)$$

Due to (21), for large values of $N - M$, we have that

$$\text{TC}_\epsilon^{\text{zf}} \approx \log(1 + \theta) \frac{(\epsilon - 1) \log(1 - \epsilon)}{\pi R^2 \theta^\alpha} \cdot (N - M + 1)^\alpha M^{1-\alpha}. \quad (33)$$

$\text{TC}_\epsilon^{\text{zf}}$ in (33) can be analytically optimized⁷ over the number of streams M by allowing $M \in (0, +\infty)$ and setting the following derivative to zero

$$\frac{\partial}{\partial M} (N - M + 1)^\alpha M^{1-\alpha} = 0. \quad (34)$$

After simple manipulations, we obtain $M_o^{\text{zf}} = (1 - \alpha)(N + 1)$. Since the constraints $M_o^{\text{zf}} \leq N$ and $M_o^{\text{zf}} \in \mathcal{Z}^+$ must also be satisfied, the optimal number of streams is

$$M_o^{\text{zf}} = \min \{ \lceil (1 - \alpha)(N + 1) \rceil, N \} \quad (35)$$

where, with a slight abuse of notation $\lceil (1 - \alpha)(N + 1) \rceil$ denotes the closest integer number to $(1 - \alpha)(N + 1)$ that maximizes (33). Note that $(1 - \alpha)(N + 1) \leq N$ holds if and only if $\alpha \geq 1/(N + 1)$ or $b \leq 2(N + 1)$, which is valid for large N as, typically, $b \leq 6$.

Setting $M = (1 - \alpha)(N + 1)$ in (33), we easily obtain that $\text{TC}_\epsilon^{\text{zf}} \propto \alpha^\alpha (1 - \alpha)^{1-\alpha} (N + 1)$, which implies that $\text{TC}_\epsilon^{\text{zf}} = \Theta(N)$. The linear scaling is the result of the appropriate choice of the number of streams such that the information rate per MIMO link is optimally traded off with the amount of interference introduced to the network. If, e.g.,

⁷Note that we can also optimize over the SIR threshold θ as in [17].

$M = 1$ or $M = N$, (32) reveals that $\text{TC}^{\text{zf}} = \Theta(N^\alpha)$ and $\text{TC}^{\text{zf}} = \Theta(N^{1-\alpha})$, respectively, i.e., the scaling is *sublinear*. This result is reminiscent of the one in [13]; the optimal contention density scales linearly in N for large N , only when the number of cancelled interferers is a fraction of N . Interestingly, the optimal value of this fraction is also $1 - \alpha$.

B. ZF-SIC (VBLAST)

Suppose that the RX employs ZF-SIC (VBLAST), i.e., it cancels out each packet that has already been decoded. The spatial diversity order corresponding to the “worst” packet, i.e., the packet that is decoded first is $N - M + 1$. Given an outage constraint ϵ on this worst stream, the maximum contention density is also given by (31). Assuming that perfect interference cancellation takes place, i.e., there is no error propagation, the transmission capacity of VBLAST is given by

$$\text{TC}_\epsilon^{\text{vb}} \approx \lambda_\epsilon^{\text{zf}} \log(1 + \theta) \sum_{m=1}^M P_u(\lambda_\epsilon^{\text{zf}}, N - m + 1), \quad (36)$$

as the spatial diversity order corresponding to each stream progressively increases as more streams are subtracted [7]. The summation term in (36) is the total throughput of all transmitted streams, with corresponding diversity orders $N - M + 1, \dots, N$, ordered from the worst to the best.

Due to the complicated nature of (36), it is not possible to determine analytically the optimum number of streams. The optimization is performed numerically in Section V.

C. DBLAST

In Sections IV-A and IV-B, a packet is transmitted on the same antenna for the duration of a slot, thereby experiencing the same fading conditions across that slot. In DBLAST [7], [18], a packet is separated into segments which are transmitted across the antennas and time, such that each segment experiences different fading conditions. The segments are then detected at the RX by ZF-SIC and, once a packet is decoded, its contribution to the received signal is subtracted. It is known that DBLAST, in conjunction with appropriate coding, approaches⁸ the outage performance of the MIMO Rayleigh channel [7], i.e., for a total transmission rate $M\mathcal{R}$, the packet outage probability is given by

$$P_o^{\text{db}} = \text{P} \left(\log \det \left(\mathbf{I}_N + \frac{1}{z} \mathbf{H} \mathbf{H}^H \right) < M\mathcal{R} \right). \quad (37)$$

In the system model we are investigating, this probability has to be evaluated over the distributions of \mathbf{H} and z . A way to approach this evaluation analytically is to recall that P_o^{db} may be upper-bounded as [18]

$$\begin{aligned} P_o^{\text{db}} &\leq \text{P} \left(\sum_{m=1}^M \log \left(1 + \frac{a_m}{z} \right) < M\mathcal{R} \right) \\ &= \text{P} \left(\prod_{m=1}^M \left(1 + \frac{a_m}{z} \right)^{\frac{1}{M}} < 1 + \theta \right) \end{aligned} \quad (38)$$

⁸In practice, DBLAST suffers from a rate loss due to initialization.

where $\{a_m\}$ are independent chi-square random variables with $a_m \sim \chi_{2(N-m+1)}^2$. We observe that, for reasonable values of θ (e.g., $\theta = 5 - 20$ dB), an outage roughly occurs when the interference power z obtains a large value. In this case, the geometric mean of $\{1 + \frac{a_m}{z}\}_{m=1}^M$ is approximately equal to the arithmetic mean, thus

$$P\left(\prod_{m=1}^M \left(1 + \frac{a_m}{z}\right)^{\frac{1}{M}} < 1 + \theta\right) \approx P\left(\frac{1}{z} \sum_{m=1}^M a_m < M\theta\right). \quad (39)$$

(The accuracy of this approximation is verified in Section V.) Moreover, $\sum_{m=1}^M a_m \sim \chi_{2N_{\text{tot}}}^2$, where

$$N_{\text{tot}} = \sum_{m=1}^M (N - m + 1) = \frac{2NM - M^2 + M}{2}. \quad (40)$$

As a result, an approximation to the success probability for DBLAST, P_s^{db} , can be obtained from (16) as follows

$$P_s^{\text{db}} \approx e^{-c(M\theta)^\alpha} + e^{-c(M\theta)^\alpha} \sum_{k=1}^{N_{\text{tot}}-1} \frac{(c(M\theta)^\alpha)^k}{k!} \sum_{n=k}^{N_{\text{tot}}-1} \frac{|\beta_k^n|}{n!}. \quad (41)$$

By Proposition 2, for $\lambda \rightarrow 0$, $P_s^{\text{db}} \approx P_u(\lambda, N_{\text{tot}})$, so, under a constraint $P_s^{\text{db}} = 1 - \epsilon$, the optimal contention density for DBLAST is

$$\lambda_\epsilon^{\text{db}} \approx -\frac{\log(1 - \epsilon)}{\pi R^2 \theta^\alpha} \frac{\Gamma(N_{\text{tot}})}{\Gamma(N_{\text{tot}} - \alpha) M^{2\alpha}} \quad (42)$$

and the respective transmission capacity is

$$\text{TC}_\epsilon^{\text{db}} \approx \log(1 + \theta) \frac{(\epsilon - 1) \log(1 - \epsilon)}{\pi R^2 \theta^\alpha} \frac{\Gamma(N_{\text{tot}}) M^{1-2\alpha}}{\Gamma(N_{\text{tot}} - \alpha)}. \quad (43)$$

From (43) and (21), for large values of N_{tot} , $\text{TC}_\epsilon^{\text{db}}$ can be approximated as

$$\text{TC}_\epsilon^{\text{db}} \approx \log(1 + \theta) \frac{(\epsilon - 1) \log(1 - \epsilon)}{\pi R^2 \theta^\alpha} \cdot 2^{-\alpha} (2N - M + 1)^\alpha M^{1-\alpha}. \quad (44)$$

As in Section IV-A, letting $M \in (0, +\infty)$ and setting the derivative of $\text{TC}_\epsilon^{\text{db}}$ with respect to M equal to zero, we obtain that $M_o^{\text{db}} = (1 - \alpha)(2N + 1)$. Under the constraints $M \leq N$ and $M \in \mathcal{Z}^+$, the optimal number of streams for DBLAST is therefore

$$M_o^{\text{db}} = \min \{[(1 - \alpha)(2N + 1)], N\} \quad (45)$$

where, as in (35), with a slight abuse of notation $\lceil (1 - \alpha)(2N + 1) \rceil$ denotes the closest integer number to $(1 - \alpha)(2N + 1)$ that maximizes (44). Note that $(1 - \alpha)(2N + 1) \leq N$ is only possible if $\alpha \geq \frac{N+1}{2N+1} > \frac{1}{2}$. This implies that, if $b \geq 4$ (which is a typical value of b for ground propagation) transmission with all antennas maximizes the transmission capacity if the network is operated in the small outage probability regime.

We now investigate how $\text{TC}_\epsilon^{\text{db}}$ scales with N . Letting $M = M_o^{\text{db}}$ in (44) (but omitting the operation $\lceil \cdot \rceil$ for simplicity), we obtain that

$$\text{TC}_\epsilon^{\text{db}} \propto \begin{cases} 2^{-\alpha} \alpha^\alpha (1 - \alpha)^{1-\alpha} (2N + 1) & \alpha \geq \frac{N+1}{2N+1} \\ 2^{-\alpha} (N + 1)^\alpha N^{1-\alpha} & \alpha < \frac{N+1}{2N+1} \end{cases} \quad (46)$$

$$\quad (47)$$

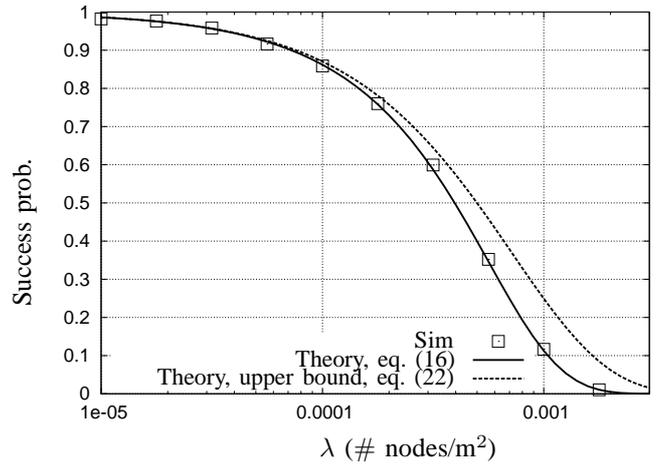


Fig. 2. Success probability vs. λ for $N = 4$ and $M = 1$ ($R = 20$ m, $b = 4$, $\theta = 6$ dB).

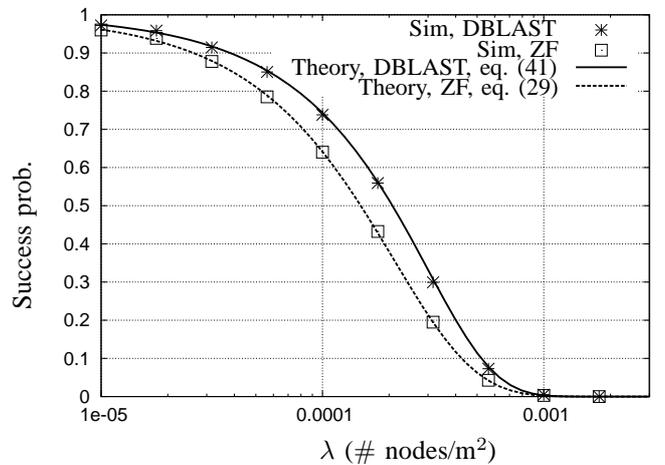


Fig. 3. Success probability vs. λ for $N = 4$ and $M = 3$ ($R = 20$ m, $b = 4$, $\theta = 6$ dB).

As a result, in both cases, $\text{TC}_\epsilon^{\text{db}} = \Theta(N)$. Finally, comparing the optimized transmission capacity of DBLAST with that of ZF, we have that, for large N and $\alpha \geq 1/2$, $\text{TC}_\epsilon^{\text{db}} \approx 2^{1-\alpha} \text{TC}_\epsilon^{\text{zf}}$, while, for $\alpha < 1/2$, $\text{TC}_\epsilon^{\text{db}} \approx \frac{2^{-\alpha}}{\alpha^\alpha (1-\alpha)^{1-\alpha}} \text{TC}_\epsilon^{\text{zf}}$. The gain in both cases is a direct consequence of the robustness of DBLAST with respect to the fading, as the information in each packet is coded and transmitted across *all* the antennas during a slot.

V. NUMERICAL RESULTS

In this section, we consider a network with default parameter values $R = 20$ m, $b = 4$, $\theta = 6$ dB. In Fig. 2, we plot the theoretical - eq. (16) - and simulated success probability as a function of the PPP density when $M = 1$ and $N = 4$. The agreement between theory and simulation confirms the validity of the analysis in Section III. We also plot the upper bound to the success probability given by (22). As shown in Proposition 2, the bound becomes tight for values of the success probability greater than 0.8 (or, as the PPP density becomes progressively smaller).

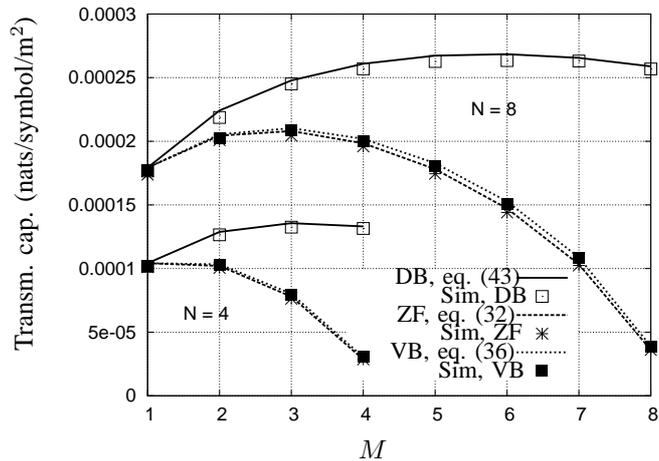


Fig. 4. Transmission capacity vs. M for $N = 4, 8$ ($\epsilon = 0.1$, $R = 20$ m, $b = 3$, $\theta = 6$ dB). The optimal number of streams for DBLAST is 3 when $N = 4$, and 6 when $N = 8$. These numbers are in accordance with (45). In the case of ZF, the optimal number of streams is 1 when $N = 4$, and 3 when $N = 8$. These numbers are also in accordance with (35). To avoid cluttering the figure, DBLAST is denoted as DB and VBLAST as VB.

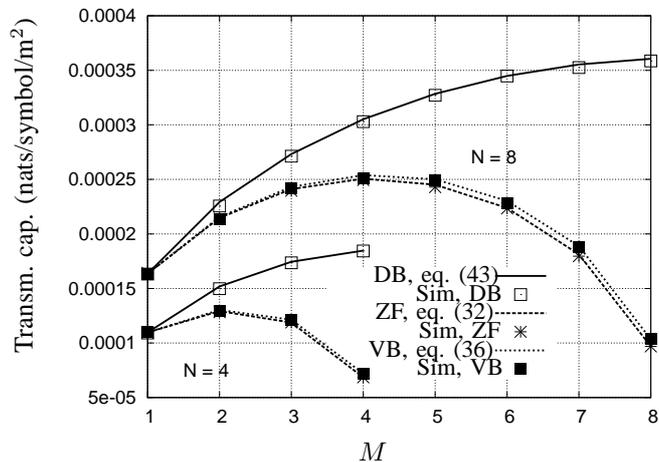


Fig. 5. Transmission capacity vs. M for $N = 4, 8$ ($\epsilon = 0.1$, $R = 20$ m, $b = 4$, $\theta = 6$ dB). To avoid cluttering the figure, DBLAST is denoted as DB and VBLAST as VB.

In Fig. 3, the theoretical and simulated success probability of ZF and DBLAST are plotted vs. the density of the PPP for a system where $M = 3$ antennas are employed in each TX. The agreement between theory and simulation is once again very satisfactory, which, in the case of DBLAST, confirms the validity of the approximations in Section IV-C.

Fig. 4 shows the dependence of the transmission capacity on the number of transmitted streams for the three MIMO techniques considered in Section IV. The total number of antennas takes two values $N = 4, 8$, the propagation exponent is $b = 3$ and a constraint $\epsilon = 0.1$ is placed on the outage probability. The DBLAST transmission scheme results in higher transmission capacity compared to VBLAST or simple ZF. Moreover, the gain between VBLAST and simple ZF is marginal, which is attributed to the fact that, with VBLAST, the maximum contention density is still determined by the subchannel with the smallest diversity order.

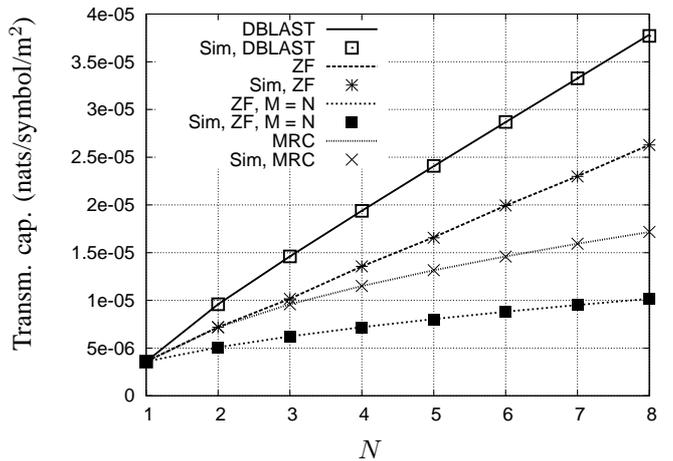


Fig. 6. Transmission capacity vs. N ($\epsilon = 0.01$, $R = 20$ m, $b = 4$, $\theta = 6$ dB). If all TX antennas are activated, the transmission capacity of ZF is lower than the transmission capacity of MRC, even though the scaling in both cases is $\Theta(\sqrt{N})$. Setting $M = N$ and $b = 4$ in (32) we can also see that $\text{TC}_\epsilon^{\text{ZF}} \approx \frac{\text{TC}_\epsilon}{\Gamma(1/2)}$, which is confirmed by the plot.

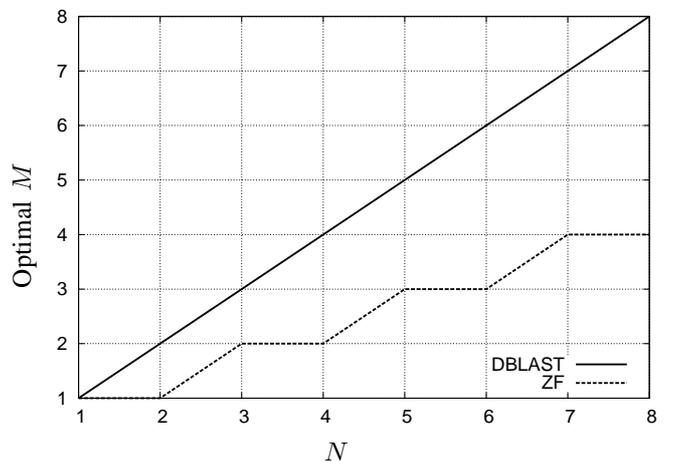


Fig. 7. Optimal number of streams vs. N ($\epsilon = 0.01$, $R = 20$ m, $b = 4$, $\theta = 6$ dB).

In Fig. 5, the propagation exponent takes the value $b = 4$. As predicted in Section IV-C, activating all the TX antennas maximizes the transmission capacity for DBLAST. In the case of ZF, the optimal number of streams is dictated by (35). Overall, in both Fig. 4 and Fig. 5, the agreement between theory and simulation is satisfactory.

In Fig. 6 and Fig. 7, the transmission capacity and the respective - optimal - number of streams are plotted vs. N for DBLAST, ZF and MRC and an outage probability constraint $\epsilon = 0.01$. As predicted in Section IV, in the case of DBLAST and ZF, and optimally selected M , the transmission capacity scales linearly in N , while, in the case of MRC, it scales as $N^\alpha = \sqrt{N}$. At $N \geq 3$, DBLAST provides a capacity gain of approximately 1.4 compared to ZF, which is in agreement with the $\sqrt{2}$ gain predicted at the end of Section IV-C. Moreover, it is observed that, for $N \leq 3$, MRC and ZF result in approximately the same transmission capacity.

VI. CONCLUDING REMARKS

In this paper, we conducted a study of a multiple-antenna single-hop random network, where the locations of the transmitters are determined according to a homogeneous PPP. Assuming channel knowledge at the RX only and that interference from concurrent transmissions is regarded as noise, we first evaluated the outage/success probability for single-antenna transmission and MRC at the RX. We then used the results and insights from this analysis in order to evaluate the outage performance and the corresponding transmission capacity of MIMO techniques such as ZF, VBLAST and DBLAST. We determined the optimum number of streams such that the transmission capacity of the network is maximized in the small outage probability regime and quantified the capacity gain of DBLAST over ZF and VBLAST.

In conclusion, our results shed light on how MIMO techniques, which are well understood in the single-user context, affect the capacity of a random wireless network. At the heart of the analysis and the resulting design guidelines lies the PPP geometric model, which allows us to take into account the randomness in the locations of the interfering TXs in the statistics of the interference power seen at the typical RX.

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