Krylov Subspace Methods in Application to WCDMA Network Optimization

Rafal Zdunek and Maciej Nawrocki

***Abstract*—Krylov subspace methods, which include, e.g. CG, CGS, Bi-CG, QMR or GMRES, are commonly applied as linear solvers for sparse large-scale linear least squares problems. In the paper, we discuss the usefulness of such methods to the optimization of WCDMA networks. We compare the selected methods with respect to their convergence properties and computational complexity, using a typical uplink model for a WCDMA network. The comparison shows that GMRES is the most suitable method for our task.**

***Index Terms*—Krylov subspace methods, WCDMA network optimization, linear solvers, CG, GMRES**

I. Introduction

Our considerations are restricted to WCDMA network optimization at the stage of layout design. In this approach, the variables of the cost function are usually expressed in terms of transmitted powers that depend on the parameters to be optimized. The parameters basically concern base stations, i.e. their number, locations, antenna azimuth and tilt as well as pilot channel powers. The details on this are given in [1]. Excluding very simplified models, the transmitted powers usually cannot be presented as analytical functions of the desired parameters. This implies the use of numerical methods for the computations of transmitted powers. Computing these powers is the most computationally intensive task in an overall optimization problem, so finding a proper (fast) method seems to be crucial. Assuming the target Signal-to-Interference (SIR) values for each link between a Base Station (BS) and a Mobile Station (MS), the transmitted powers can be computed from the system of linear equations:

$$\boldsymbol{A}\boldsymbol{p} = \boldsymbol{b}, \quad (1)$$

where $\boldsymbol{A} =\in I\!R^{K\times K}$ is the system matrix of coefficients that depend on the link gains, othogonality factors (for downlink) and target SIR values, $\boldsymbol{p} \in I\!R^{K}$ is the vector of unknown transmitted powers, $\boldsymbol{b} \in I\!R^{K}$ is the noise vector. The aim is to find a possible best estimation of the vector $\boldsymbol{p}$ at the least computational cost. It should be noted that the task is very challenging since the system (1) can be very large (even after applying the dimension reduction technique [2], [3]) and such estimations must be repeated many times to provide many Monte Carlo (MC) samples used in static simulators for network planning and optimization [4], [5]. The system (1) has rather good numerical properties (square, consistent, well-conditioned), and hence many linear solvers can be used in our application. Nevertheless, not all the methods have the same convergence properties and computational complexity, thus there is a need to study the usefulness of these methods to our task. The problem has been already tackled for in our previous works [1], [6], [7], where we compared the Gaussian elimination and some iterative methods such as Jacobi, Gauss-Seidel, Successive Over-Relaxation (SOR), and Conjugate Gradient Square (CGS). Some numerical results from [6], [7] will be reminded here. Finally, we concluded in [6] that the Gauss-Seidel and CGS gave the best results. Since the CGS belongs to a class of Krylov subspace methods, we decided to continue our tests with respect to the Krylov subspace methods which we shortly present in Section II. The comparison results are presented in Section III, and finally some concluding remarks are given in Section IV.

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II. Krylov Subspace Methods

Krylov subspace methods are widely applied to solve large-scale linear systems arising in many areas of science, especially to solve discretized Partial Differential Equations (PDE) [8], [9]. Due to their low computational cost, the methods can be also useful in the optimization of WCDMA networks. A short survey of the Krylov subspace methods that are used in our experiments is given below.

- **CGLS**
  The first version of the Conjugate Gradients (CG) method was proposed by Hestenes and Stiefel [10], and it is commonly used for solving symmetric linear systems. It iteratively minimizes the gradient of a quadratic objective function with gradient updates derived from orthogonal directions. Since in our application the symmetry condition is not met, the CG method is applied to the normal equations. In the literature, such method is known as CGLS and it may be found in many implementations. We used the Hansen's implementation [11].
- **CGS**
  The Conjugate Gradient Square (CGS) method was invented by Sonneveld [12] and it involves the CG scheme. In contrary to the CG, it can be used to non-symmetric systems. Moreover, it is not sensitive to so-called the serious breakdown that may occur in the CG.
- **BiCG**
  The Bi-Conjugate Gradient (BiCG) method, proposed by Fletcher [13], belongs to a group of bi-orthogonal methods and extends the standard CG method to non-symmetric, large and sparse systems of linear equations. Hence, it may be suitable for our application.

TABLE I
COMPUTATIONAL COST OF ONE ITERATIVE STEP FOR THE ANALYZED METHODS. THE SUBSCRIPTS m, d, a, s DENOTE ELEMENTARY MULTIPLICATIVE, DIVISION, ADDITION, AND SUBTRACTION OPERATIONS. THE SUBSCRIPT f STANDS FOR A FUNCTION EVALUATION (SQUARE ROOTING OR POWERING).

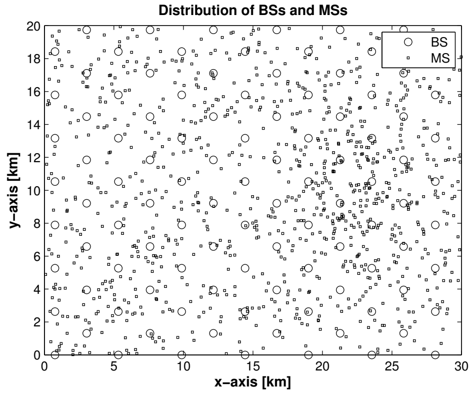
Method	Computational cost of one iteration
CGLS	$(5K^2+6K)_{m/d}+(5K^2+7K)_{a/s}$
CGS	$(2K^2+10K)_{m/d}+(2K^2+12K)_{a/s}$
BiCG	$(4K^2+9K)_{m/d}+(5K^2+10K)_{a/s}$
BiCGSTAB	$(6K^2+12K)_{m/d}+(6K^2+14K)_{a/s}$
QMR	$(3K^2+14K)_{m/d}+(3K^2+14K)_{a/s}+(2K+2)_f$
GMRES	depends on many factors (sparsity)

- **BiCGSTAB**
  The BiConjugate Gradients Stabilized (BiCGSTAB) method was developed by Van der Vorst [8], [9]. The BiCGSTAB differs from the CGS only with the way of computing a residual vector. It is reported in [8] that the BiCGSTAB has better convergence properties due to local minimization of successive updates for the residual vector. The curve of the l_2 norm of the residual vector is smoother and steeper than for the CGS. Unfortunately, some perturbations in convergence or even a serious breakdown of an iterative process may occasionally happend, especially if the system matrix has complex eigenvalues.
- **QMR**
  The Quasi-Minimal Residual (QMR) method that was designed by Freund and Nachtigal [14] uses the similar assumptions as the BiCG but considerable difference exists in the residual smoothing technique. Its highest advantage is a numerical stability, i.e. it avoids the case of serious breakdown. There are many implementations of the QMR [9], [14]. In the experiments we used the implementations given in MATLAB 7.0.
- **GMRES**
  The GMRES method was proposed by Saad and Schultz [15] for solving linear least squares problems with non-symmetric matrices without a necessity of creating the normal equations. In the experiments we used the MATLAB implementation where the Gram-Schmidt orthogonalization is obtained with the Givens rotations.

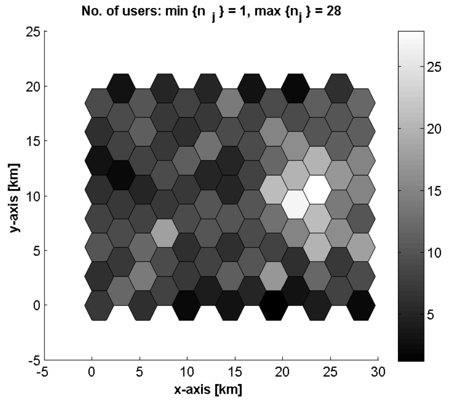
The roughly estimated computational costs of all the algorithms used in our experiments are given in Table. 1. The computational cost for the GMRES is not easy to estimate because it depends on the system matrix used. For a sparse matrix, the cost is considerably lower than for a dense matrix because the related number of the involved Givens rotations is much smaller. In our application, the system matrix may be very sparse if a large network is analyzed (without using the dimension reduction technique [2], [3]).

III. NUMERICAL RESULTS

The experiments demonstrating the efficiency of the analyzed methods are performed for a randomly selected MC snapshot in uplink transmission with both omnidirectional



(a)

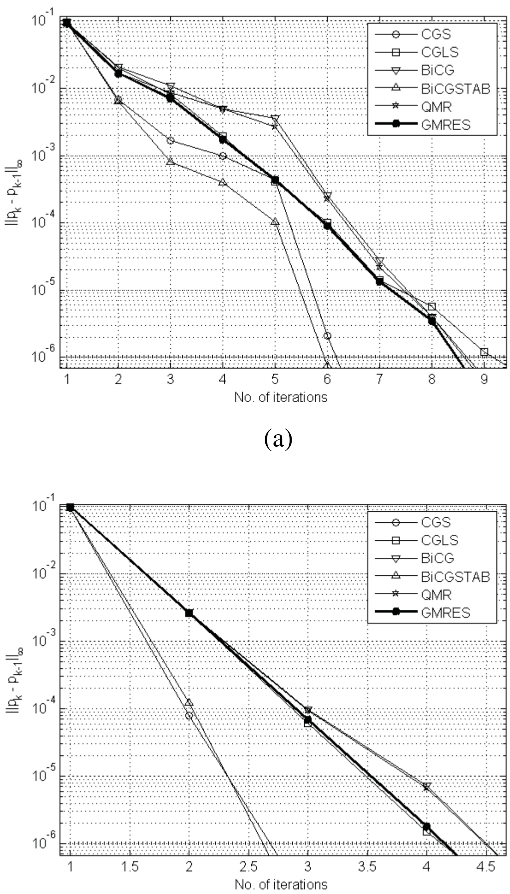


(b)

Fig. 1. (a) Layout of BSs and MSs; (b) The numbers of users assigned to each cell.

antennas and Smart Antennas (SA). Typically, we assume 1000 users randomly distributed in 104 cells with a mixture of uniform and skrew-Gaussian distributions. Hence, we have $\boldsymbol{A} \in I\!R^{1000\times 1000}$, $K = 1000$ and $M = 104$. In our approach, we assume that the analyzed network is not overloaded. For the overloaded case, some values of the target SIR vector should be decreased, which can be done with many techniques, e.g. with the one described in [3]. The layout of BSs and MSs is presented in Fig. 1(a). The geometry of the tested area and the number of the users in each cell are shown in Fig. 1(b). Half of the users work with a voice service ($R_b = 12.2$kbps), and the other half with a data service ($R_b = 64$kbps).

For this snapshot and traditional antennas (omnidirectional): $\max_i\{|\lambda_i(\boldsymbol{A})|\} = 2.1\times 10^{-7}$ and $\min_i\{|\lambda_i(\boldsymbol{A})|\} = 8.7\times 10^{-13}$, and for the SA: $\max_i\{|\lambda_i(\boldsymbol{A}_{(SA)})|\} = 2.1\times 10^{-6}$ and $\min_i\{|\lambda_i(\boldsymbol{A}_{(SA)})|\} = 9.2\times 10^{-12}$. Hence, the convergence of the Krylov subspace method is definitely guaranteed [8], [9], [12]–[15]. All the iterative algorithms are run until the stopping criterion $\boldsymbol{e}^k = ||\boldsymbol{p}^k - \boldsymbol{p}^{k-1}||_\infty \geq \epsilon$ is met, where for arbitrary $\boldsymbol{u}$: $||\boldsymbol{u}||_\infty = \max_i\{u_i\}$, and ϵ is a small positive number. We assume that the solution should be computed with the accuracy up to the fifth significant digit, thus $\epsilon = 16^{-6}$. The plots of $\boldsymbol{e}^k$ versus iterations are illustrated in Fig. 2(a) and Fig. 2(b) for the cases of traditional antennas and SAs, respectively.



(a)

(b)

Fig. 2. History of error e^k versus iterations for: (a) traditional antennas, (b) SA.

The dashed horizontal lines in Fig. 2 mark the error level of 10^{-6} at which the iterative process is stopped. It follows from Fig. 2(a) that this level or lower is reached by the CGS, CGLS, BiCG, BiCGSTAB, QMR and GMRES after running 7, 10, 9, 6, 9 and 9 iterations, respectively. For SAs (see Fig. 2(b)), this level is reached within 3, 5, 5, 3, 5 and 5 iterations for respective methods. In [6] the Richardson, Jacobi, Gauss-Seidel, SOR methods stopped at the same error level after performing 36, 50, 29, 14 iterations for traditional antennas, and 15, 4, 3, 5 iterations for SAs, respectively. All the discussed methods have been applied to the preconditioned version of the system (1), where the right-hand preconditioning was applied as in [1], [6], [7].

To simplify the comparison analysis, let us drop the notation of the kind of arithmetic operations. First, let us consider traditional antennas. Thus, it follows from Table I that the computational cost of performing 7 iterations with the CGS is about $28K^2 + 154K$ arithmetic operations. For the CGLS, BiCG, BiCGSTAB and QMR we have: $100K^2 + 130K$, $91K^2+171K$, $72K^2+156K$ and $54K^2+252K+18K$, respectively, where additional $18K$ in QMR means the cost related to the function evaluation, which may be quite expensive but dependent on the software and hardware used. To remind, we got $72K^2 + 108K$, $100K^2 + 150K$, $87K^2$, and $44K^2 + 16K$ for the preconditioned Richardson, Jacobi's, Gauss-Seidel, and SOR methods. A similar analysis for the case of SAs gives the following rough estimations of the costs: $32K^2 + 46K$, $8K^2 + 12K$, $9K^2$, $16K^2 + 6K$, $14K^2 + 67K$, $50K^2 + 65K$, $45K^2 + 95K$, $36K^2 + 78K$, and $30K^2 + (140 + 10)K$ for the corresponding methods: Richardson, Jacobi, Gauss-Seidel, SOR, CGS, CGLS, BiCG, BiCGSTAB, and QMR. Because the estimation of the cost for GMRES is not so easy, we compare this method only with respect to the elapsed time of performing 10 iterations in the same computational and hardware environment (MATLAB 7.0).

The elapsed time [in seconds] measured in MATLAB is given in Table II where we compare the methods applied to the problems of different scales. The first two columns refer to the small-scale problem that occured after applying the dimension reduction technique ([2], [3]) to the snapshot described above ($M = 104$, $K = 1000$). Thus, our system matrix is reduced to the size 104 by 104. Since in real applications much bigger problems must be resolved, we analyze a bigger case – the snapshot with 300 cells and 3000 users – without using the dimension reduction technique but with the above-mentioned preconditioning. The elapsed times are given in the last two columns. Note that the measured time is exemplary and in each snapshot it may be slightly different due to the difference in properties of the system matrix.

IV. Conclusions

Comparing the estimations of the computational costs, we can conclude that the Gauss-Seidel method is the most promising, especially for the SA case. For the traditional antennas, the CGS is the fastest, and then the SOR.

However, with reference to Table II, we can conclude that for large-scale problems the GMRES is the fastest algorithm. Thus, for the analysis of a large network (with many BSs), the GMRES should be used in a static simulator. For small-scale problems, especially for a small number of BSs, the Gauss-Seidel and CGS are optimal.

Acknowledgments

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References

[1] M. J. Nawrocki, M. Dohler, and A. H. Aghvami, Eds., *Understanding UMTS Radio Network Modelling, Planning and Automated Optimisation: Theory and Practice*. John Wiley and Sons, 2006.

[2] L. Mendo and J. M. Hernando, "On dimension reduction for the power control," *IEEE Trans. On Communications*, vol. 49, no. 2, pp. 243–248, 2001.

[3] R. Zdunek and M. J. Nawrocki, "Improved modeling of highly loaded UMTS network with nonnegative constraints," in *IEEE 17th International Symposium on Personal, Indoor and Mobile Radio Communications (PIMRC 2006)*, Helsinki, Finland, September 2006.

[4] J. Laiho, A. Wacker, and T. Novosad, *Radio Network Planning and Optimization for UMTS*. Chichester: John Wiley and Sons, 2002.

[5] A. Wacker, J. Laiho-Steffens, K. Sipila, and M. Jasberg, "Static simulator for studying WCDMA radio network planning issues," in *Proc. IEEE Vehicular Technology Conference*, Houston, Texas, USA, May 1999, pp. 2436–2440.

TABLE II
ELAPSED TIME [IN SECONDS] OF PERFORMING 10 ITERATIONS WITH DIFFERENT ALGORITHMS AND FOR DIFFERENT SIZE OF THE ANALYZED NETWORK EQUIPPED WITH TRADITIONAL (T) AND INTELLIGENT (SMART) ANTENNAS.

Problem/Method	$K = 1000$ $M = 104$ $\boldsymbol{A} \in I\!R^{104\times104}$ (T)	$K = 1000$ $M = 104$ $\boldsymbol{A} \in I\!R^{1000\times1000}$ (SMART)	$K = 3000$ $M = 300$ $\boldsymbol{A} \in I\!R^{3000\times3000}$ (T)	$K = 3000$ $M = 300$ $\boldsymbol{A} \in I\!R^{3000\times3000}$ (SMART)
Richardson	0.04	0.13	1.056	1.101
Jacobi	0.01	0.13	1.072	1.081
Gauss-Seidel	0.01	0.231	1.952	1.923
SOR	0.02	0.311	2.943	2.824
CGLS	0.03	0.12	1.121	0.991
BiCG	0.06	0.211	1.562	1.523
BiCGSTAB	0.088	0.41	2.053	2.403
CGS	0.011	0.257	1.572	1.701
QMR	0.091	0.241	1.592	1.643
GMRES	0.10	0.15	0.691	0.771

[6] R. Zdunek, M. J. Nawrocki, M. Dohler, and A. H. Aghvami, "Application of linear solvers to UMTS network optimization without and with smart antennas," in *IEEE 16th International Symposium on Personal, Indoor and Mobile Radio Communications (PIMRC 2005)*, vol. 4, Berlin, Germany, September 11–14 2005, pp. 2322–2326.

[7] R. Zdunek and M. J. Nawrocki, "On linear solvers in applications to WCDMA network optimization," in *Proc. National Conference on Radio-communication, Radio and Television (KKRRiT)*, Krakow, Poland, June 15–17 2005, pp. 77–80.

[8] G. H. Golub and H. A. V. der Vorst, "Closer to the solution: Iterative linear solvers," in *The State of the Art in Numerical Analysis*, I. Duff and G. Watson, Eds. Clarendon Press, Oxford, 1997, pp. 63–92.

[9] Y. Saad and H. A. V. der Vorst, "Iterative solution of linear systems in the 20-th century," *Journal of Computational and Applied Mathematics*, vol. 123, no. 1–2, pp. 1–33, 2000.

[10] M. R. Hestenes and E. Stiefel, "Method of conjugate gradients for solving linear systems," *J. Res. Nat. Bur. Standards*, vol. 49, pp. 409–436, 1952.

[11] P. C. Hansen, "Regularization tools version 4.0 for matlab 7.3," *Numerical Algorithms*, vol. 46, pp. 189–194, 2007.

[12] P. Sonneveld, "CGS: A fast lanczos-type solver for nonsymmetric linear systems," *SIAM J. Sci. Statist. Comput.*, vol. 10, pp. 36–52, 1989.

[13] R. Fletcher, "Conjugate gradient methods for indefinite systems," in *Numerical Analysis*, ser. Lecture Notes Math., G. Watson, Ed. Berlin-Heidelberg-New York: Springer-Verlag, 1976, vol. 506, pp. 73–89.

[14] R. W. Freund and N. M. Nachtigal, "QMR: a quasi-minimal residual method for non-hermitian linear systems," *Numer. Math.*, vol. 60, pp. 315–339, 1991.

[15] Y. Saad and M. H. Schultz, "GMRES: a generalized minimal residual algorithm for solving nonsymmetric linear systems," *SIAM J. Sci. Statist. Comput.*, vol. 7, pp. 856–869, 1986.

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