# New Tailbiting Convolutional Codes over Rings

Piotr Remlein and Dawid Szłapka

***Abstract*—In this paper a method of using convolutional codes over rings for packet data transmission over additive white gaussian noise (AWGN) channel is proposed. The tailbiting method is generalized and applied to convolutional codes based on ring of integers modulo-M. The codes were named tailbiting codes over ring (TBR). This paper presents a method to desing TBR codes obtained by the concatenation of feedback convolutional encoder over ring and M-QAM modulator. The paper describes how a systematic ring convolutional encoder with feedback can obtain the same starting and ending state. The best TBR codes with different number of encoder states for 16-QAM modulated symbol sequences of varying lengths are tabulated.**

***Index Terms*—Convolutional codes over rings, tailbiting codes**

## I. Introduction

PACKET data transmission schemes are often used in wireless telecommunication systems. The convolutional codes are used in such systems as an efficient and powerful class of error correcting codes [1]. To be able to use convolutional codes (of rate $R = k/n$ and $m$ memory elements) in the packet transmission, we must convert these codes to block codes. There are some methods for this conversion. One of such methods is called tailbiting. In this method, no additional bits are appended to the codeword to drive the encoder to a known state [2]. The encoder starts and finishes the encoding process in the same state but this state is not known by the decoder. In the paper [3] it is shown that the turbo-codes generally provide the best error rate performance for long blocks (over 150 bits), but for short blocks (under 150 bits) the tailbiting convolutional codes provide the best performance. The motivation for investigating the ring convolutional codes was to explore a natural relation between M-ary modulation and codes over the rings of integers modulo-M [4]. Up to now, the best tailbiting codes with the greatest minimum Hamming distance were published in the literature [2], [5], [6]. In case of search for the best convolutional codes for the signals transmitted over the AWGN channel, the quality the criterion is Euclidean distance [1]. In this article we assumed the Euclidean distance as a parameter to estimate the quality of TBR codes. To find the best TBR codes, one can search the full space of the codes. Such method gives the certainty that the found codes are the best. The fault of this method is the exponentially growing complexity with the growing number of memory cells of the encoder and the number of its inputs.

In this paper we analysed the tailbiting codes over rings encoded by systematic ring convolutional encoders with feedback. We present the results of the search for the best convolutional encoders over ring modulo-4 with code rate $R = 1/2$ used with 16-QAM modulation, with labeling as in [7]. We found new TBR codes for 16-QAM modulation with the best Euclidean distance.

This paper is organised as follows. Section 2 describes the procedure of encoding TBR by using the systematic convolutional encoders with feedback. In Section 3 we present the results of computer search for the best TBR codes. Finally, Section 4 gives the concluding remarks.

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## II. Tailbiting Codes over Rings – Encoding Method

In this article we generalize the tailbiting method onto convolutional codes over rings of integers modulo-m [2], [3] and we name the resulting codes tailbiting convolutional codes over rings (TBR). In the proposed method we encode and decode a block of N (M-ary) symbols without a known tail, thus keeping the effective rate of transmission equal to the code rate. This is done by letting the encoder start and end in the same state, unknown for the decoder. The encoding procedure to achieve this is not difficult if the structure of the encoder is feedforward. In this case, the starting state depends on the m last information symbols in the transmited packet, where m is the number of memory cells in the encoder. In case of convolutional encoder with feedback Fig. 1, the starting state depends on all the information symbols in the packet. Finding the initial state wherein the encoder should start encoding and – after $N$ symbol intervals – end encoding in the same state is complex. One of the methods for finding this initial state was proposed in [8] and extended for multilevel codes in [9].

In Fig. 1, we show the realization of the systematic feedback convolutional encoder over ring of integers modulo-M [4], [6] with the code rate $R = k/n, n = k + 1$.

At time $t$, the information vector $U_t$ with M-ary elements $u_t^{(i)}$ belonging to the ring $Z_M = 0, 1, 2, ..., M - 1, (\Re = Z_M)$ inputs the encoder.

$$U_t = (u_t^{(1)}, u_t^{(2)}, ..., u_t^{(k)}) \quad (1)$$

The convolutional encoder produces a coded sequence of symbols which belong to the same ring $Z_M$

$$V_t = (v_t^{(1)}, v_t^{(2)}, ..., v_t^{(n)}) \quad (2)$$

where $n = k + 1$.

The coefficients in the encoder Fig. 1 are taken from the set $0, ..., M - 1$. The memory cells are capable of storing ring elements. Multipliers and adders perform multiplication and addition, respectively, in the ring of integers modulo-M.

The encoding process can be described as mapping of the information vector (1) into the encoded vector (2)

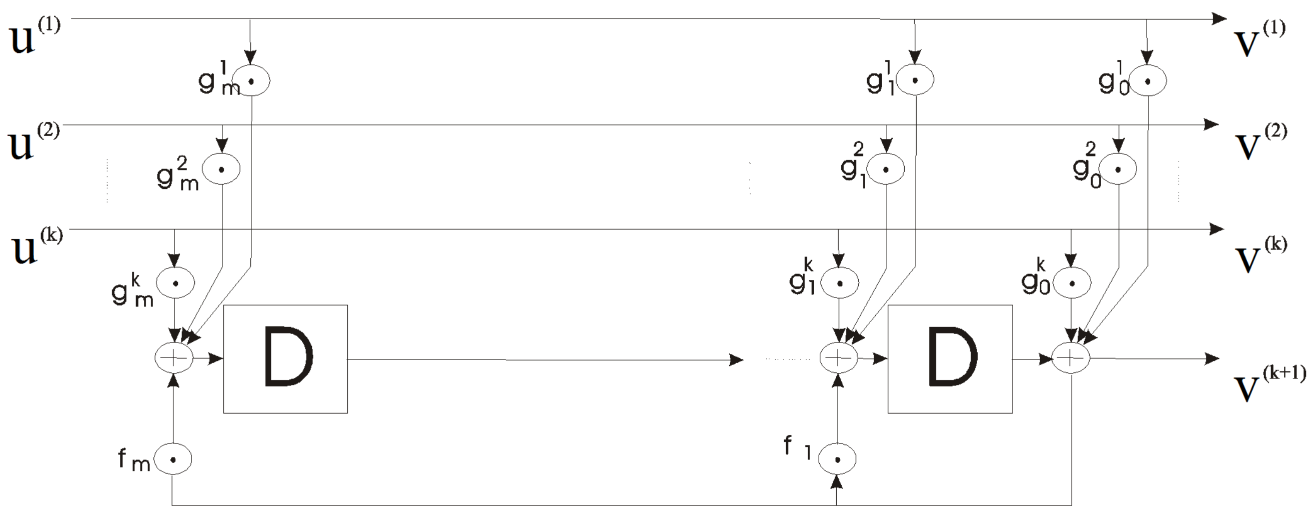


Fig. 1. Systematic feedback convolutional encoder over ring of integers modulo-M.

$$V_t = U_t G \tag{3}$$

where $G$ denotes the generator matrix of the encoder [8].

The state of the encoder at time t is determined by the content of memory elements

$$X_t = (x_t^{(1)}, x_t^{(2)}, ..., x_t^{(m)})^T, \tag{4}$$

where $m$ is the number of encoder memory elements.

In case of packet transmission without tail, where the convolutional encoders with feedback are utilized, we have to calculate the initial state $X_0$ that must be the same as the find state $X_N$ of the encoder after $N$ cycles. This is not quite easy. To find this starting state, we used the method proposed in [8]. The correct starting state can by calculated using the state space representation. The state of the encoder in time $t+1$ can be described as:

$$X_{t+1} = AX_t + BU_t^T, \tag{5}$$

where $A$ is the $(m \times m)$ state matrix which defines connections between memory elements, $B$ is the $(m \times k)$ control matrix which defines connections between encoder inputs and memory elements.

The vector $V_t$ at the encoder output in time $t$ can be described as in [8]:

$$V_t^T = CX_t + DU_t^T, \tag{6}$$

where: $C$ is the $(n \times m)$ observation matrix which defines connections between encoder outputs and memory elements, $D$ is the $(n \times k)$ transition matrix which defines connections between encoder entries and outputs.

In the paper [8] it was also shown that the state $(X_t)$ in time $t$, of the systematic convolutional encoder with feedback can be described as the superposition of two vectors $X_t^{[zi]}$ and $X_t^{[zs]}$ which define the ending state of the encoder

$$X_t = X_t^{[zi]} + X_t^{[zs]} \tag{7}$$

where $X_t^{[zi]}$ is the vector which defines the encoder state achieved after $t$ cycles if the encoding process started in state $X_0$ and all inputs symbols are zero, $X_t^{[zs]}$ is the vector which defines the encoder state achieved after $t$ cycles if the encoding stared in the all zero state $(X_0 = 0)$ and the information symbol sequence is encoded.

From the equations (5) and (7) we can write that:

$$X_t = X_t^{[zi]} + X_t^{[zs]} = A^t X_0 + \sum_{\tau=0}^{t-1} A^{(t-1)-\tau} BU_\tau^T. \tag{8}$$

If we assume that the state in time $t = N$ is equal to the initial state $X_0$, we obtain from (8):

$$(I_m - A^N)X_0 = X_N^{[zs]}, \tag{9}$$

This equation can be written for convolutional encoders over ring $\Re = Z_M$ as:

$$(I_m + A^N)X_0 = X_N^{[zs]}, \tag{10}$$

where $I_m$ is the $(m \times m)$ identity matrix. As it is seen from (10), we can calculate the correct initial state $X_0$ of the encoder if the matrix $(I_m + A^N)$ is invertible.

The matrix $A$ from equation (10) for the systematic convolutional encoder with feedback is described as [8], [9]:

$$A = \left[\begin{array}{cccc|c} 0 & \cdots & 0 & & f_m \\ 1 & & & & f_{m-1} \\ & \ddots & & & \vdots \\ & & 1 & & f_1 \end{array}\right] \tag{11}$$

Using the mathematical relations (9) and (10), obtained above we can describe the encoding process for TBR codes as follows: at first, we have to calculate the vector $X_N^{[zs]}$ for a given information data packet. Accordingly, the encoder starts in the all zero state. All the $N \cdot k$ information symbols are encoded but the output symbols are ignored. After $N$ cycles the encoder will be in the state $X_N^{[zs]}$. Then, form (10) we can calculate the correct initial state $X_0$, the encoder can start the proper encoding process and a valid codeword results. After $N$ cycles the encoder ends its work, reaches the state which is the same as its starting state.

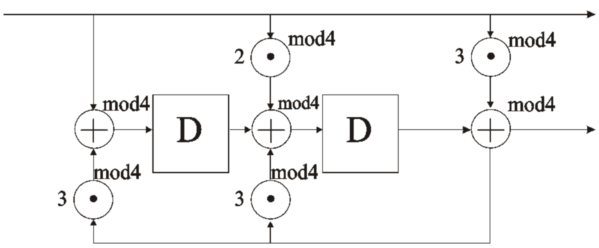


Fig. 2. Encoder of the convolutional code $G(D) = \left[ \begin{array}{cc} 1 & \frac{3+2D+D^2}{1+3D+3D^2} \end{array} \right]$ from the example.

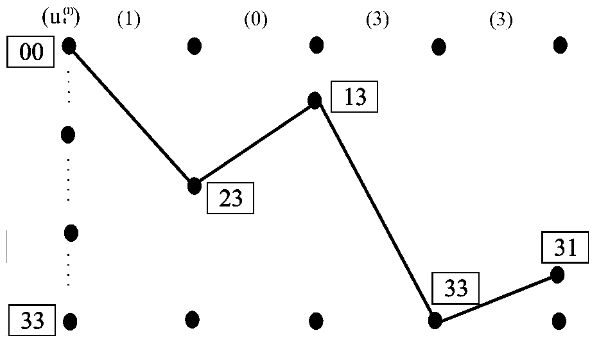


Fig. 3. Tree diagram when the zero state response is obtained $X_4^{[zs]}$.

Following this description, we show an example of TBR encoding procedure with feedback systematic convolutional encoder over ring $Z_4$.

*1) Example:* A packet of four symbols is encoded. The symbols belong to the ring $Z_4$. The encoder is a systematic convolutional encoder over ring $Z_4$ with feedback, with code rate $R = 1/2$ and two memory elements $m = 2$. In Fig. 1 we show the structure of this encoder. We encode the information block $U = (U_0, U_1, U_2, U_3) = (1, 0, 3, 3)$. The state matrix is given as $A = \left[ \begin{array}{cc} 0 & 3 \\ 1 & 3 \end{array} \right]$. Therefore, $N = 4$, $k = 1$, and from equation (9) we can calculate $\left( I_2 - \left[ \begin{array}{cc} 0 & 3 \\ 1 & 3 \end{array} \right]^4 \right) X_0 = X_4^{[zs]}$. From this formula we obtain: $X_0 = \left[ \begin{array}{cc} 2 & 1 \\ 3 & 3 \end{array} \right] X_4^{[zs]}$. Therefore, we have to calculate the state $X_4^{[zs]}$.

From Fig. 2 we can see that this state is equal to $(3, 1)^T$ and the correct state from which we must start the encoding process is equal to $X_0 = \left[ \begin{array}{cc} 2 & 1 \\ 3 & 3 \end{array} \right] \left[ \begin{array}{c} 3 \\ 1 \end{array} \right] = \left[ \begin{array}{c} 3 \\ 0 \end{array} \right]$. From Fig. 3 we can see that, if we start to encode the sequence U from state $(3, 0)^T$, then after $N = 4$ cycles we reach the same state and obtain valid codeword $V = (13, 02, 31, 30)$.

## III. Search Results

In this section we present the results of computer search for the best tailbiting codes over rings modulo-M for transmission over AWGN channel. As the quality criterion we take the minimum Euclidean distance $d_{e\_min}$. We compute the minimum Euclidean distance as the minimum distance over all pairs of distinct codewords [10]. Each coded sequence must be compared to all the other coded sequences. The codes were generated by the feedback systematic convolutional encoder

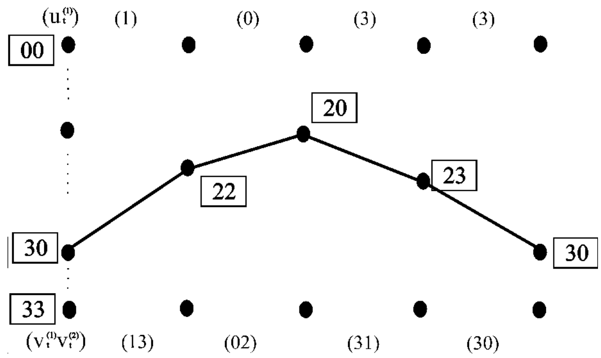


Fig. 4. Tree diagram for proper encoding process for tailbiting codes over ring $Z_4$.

over ring. An exhaustive search was used to find TBR codes in Fig. 4. The object of search in this article were tailbiting codes over ring $Z_4$, generated by concatenation of the systematic encoders with feedback with code rate $R = 1/2$ and 16-QAM modulator. The found encoders have $m$ memory cells, $S$ states and $k$ inputs. $N$ denotes the length of the input symbol sequence of $k$ information bits per symbol. For codes over ring, feedback coefficients $f_0 \sim f_m$ and the coefficients in the systematic branches $g_0^k \sim g_m^k$ are written as a sequence of decimal numbers.

The coefficients equal to zero at the beginning of the sequence are skipped in the description. All TBR codes over ring found for 16-QAM are presented in Table I. We found the best TBR codes for encoders with 16, 64 and 256 states. All of these TBR codes are the new codes that have not been published yet.

## IV. Conclusion

In this paper we generalized the tailbiting techniques onto the tailbiting codes over rings of integers modulo-M. We described how the systematic ring convolutional encoder with feedback can have the same starting and ending state. We presented the search results of the best tailbiting codes over ring $Z_4$ for the transmission over AWGN channel. As the optimization criterion of the we took the Euclidean distance.

A table of the best new tailbiting convolutional codes over ring $Z_4$ with rate $R = 1/2$ for 16-QAM modulation was obtained by computer search. All TBR codes shown in Fig. 4 have not been presented in the literature known to the authors.

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TABLE I
TAILBITING CODES OVER RING $Z_4$ WITH CODE RATE R=1/2 FOR 16-QAM MODULATION (NATURAL LABELING AS IN [7]).

| $S$ | 16 TBR | | 64 TBR | | 256 TBR | |
|---|---|---|---|---|---|---|
| $N$ | $f,g$ | $d_{e\_min}$ | $f,g$ | $d_{e\_min}$ | $f,g$ | $d_{e\_min}$ |
| 4 | 130,100 | 12,000 | 1300,1000 | 12,000 | 11100,10000 | 12,000 |
| 5 | 113,210 | 14,128 | 1130,2100 | 14,128 | 10132,10000 | 14,128 |
| 6 | 102,111 | 14,141 | 1121,1100 | 14,828 | 12330,11000 | 14,828 |
| 7 | 111,123 | 16,944 | 1312,1313 | 16,970 | 11331,20110 | 16,970 |
| 8 | 111,221 | 16,970 | 1121,120 | 18,129 | - | - |


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