Modeling Step Index Fiber to Soliton Propagation

Tomasz Kaczmarek

***Abstract*—Step index fiber modeling process is carried out through numerical solving of eigenvalue equation to calculate propagation constant for fundamental mod. Input data in the process is only index of refraction calculated from Sellmeier dispersive formula for appropriate mol percentage doping of germanium dioxide in silica glass fiber. Output data in the modeling process is optimal value of the normalized frequency, which guarantees that single mode operation region is equal to bright soliton propagation region. Final verification of the process is soliton generation up to sixth-order inside such modeled fiber. In this end nonlinear Schödinger equation is solved numerically for initial condition of hyperbolic secant form. Maximization of single mode operation and bright soliton propagation region is essential in wavelength division multiplexing technique.**

***Index Terms*—eigenvalue equation, nonlinear Schödinger equation, solitons**

I. INTRODUCTION

THE word soliton refers to special kinds of wave packets that can propagate undistorted over long distances. In the context of optical fibers solitons have found practical applications in the field of fiber-optic communications. Solitons results from a balance between group-velocity dispersion and self-phase modulation, both of which can be calculated in effect of step index fiber modeling process.

Propagation of soliton in single-mode optical fiber is described by the nonlinear Schrödinger equation [1]–[4]

$$j\frac{\partial A}{\partial z}-\frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2}+\gamma\left|A\right|^2 A=0, \tag{1}$$

where A is the slowly varying envelope of the pulse, γ is nonlinear parameter of the fiber, β_2 is group velocity dispersion, z and T are spatial and time variable, respectively. Group velocity dispersion expressed in ps^2/km is defined as the second derivative of mode propagation constant β with respect to frequency ω i.e. $\beta_2 = \mathrm{d}^2\beta/\mathrm{d}\omega^2$, and is related to dispersion parameter D expressed in $ps/(km \cdot nm)$ through the relation $D = -2\pi c\beta_2/\lambda^2$ where c is the speed of light in vacuum. Nonlinear parameter is defined as follows [1], [4]

$$\gamma=\frac{n_{NL}k}{A_{eff}}, \tag{2}$$

where n_{NL} is nonlinear refractive index, A_{eff} is known as effective core area. For pulses as short as 1 ps and in case of single mode fiber, which core is made of silica glass doped by germanium dioxide, value of n_{NL} is approximately equal to $n_{NL} = 2.2 \cdot 10^{-20} m^2/W$ [1]. Effective core area is related to the transverse component of electric field vector E_0 and effective core radius ω_{eff} through the relations [1], [4]

$$A_{eff}=\frac{2\pi\left[\int\limits_0^\infty \left|E_0\left(r\right)\right|^2 rdr\right]^2}{\int\limits_0^\infty \left|E_0\left(r\right)\right|^4 rdr}=\pi\omega_{eff}^2, \tag{3}$$

where r is radial coordinate in the cylindrical coordinate system. Absolute value of E_0 is related to the transverse components of electric field vector E_r and E_ϕ through well known formula $|E_0| = (|E_r|^2 + |E_\phi|^2)^{1/2}$. The transverse components are determined by the use of axial component of electric E_z and magnetic H_z field vectors through the following relations [3], [5], [6]

$$E_{r1}=\frac{-j}{\chi^2}\left(\beta\frac{\partial E_{z1}}{\partial r}+\frac{\omega\mu_0}{r}\frac{\partial H_{z1}}{\partial\phi}\right), \tag{4}$$

$$E_{\phi1}=\frac{-j}{\chi^2}\left(\frac{\beta}{r}\frac{\partial E_{z1}}{\partial\phi}-\omega\mu_0\frac{\partial H_{z1}}{\partial r}\right), \tag{5}$$

$$H_{r1}=\frac{-j}{\chi^2}\left(\beta\frac{\partial H_{z1}}{\partial r}-\frac{\omega\varepsilon_0 n_1^2}{r}\frac{\partial E_{z1}}{\partial\phi}\right), \tag{6}$$

$$H_{\phi1}=\frac{-j}{\chi^2}\left(\frac{\beta}{r}\frac{\partial H_{z1}}{\partial\phi}+\omega\varepsilon_0 n_1^2\frac{\partial E_{z1}}{\partial r}\right), \tag{7}$$

for the core. In case of cladding subscript 1 should be changed to 2 and, moreover, variable χ^2 should be replaced with $-\sigma^2$. Equations from (4) to (7) are essential for computing an average power curried by the core [5], [6]

$$P_1=\pi\int\limits_0^a\left(E_{r1}H_{\phi1}^*-E_{\phi1}H_{r1}^*\right)rdr, \tag{8}$$

and cladding [5], [6]

$$P_2=\pi\int\limits_a^{+\infty}\left(E_{r2}H_{\phi2}^*-E_{\phi2}H_{r2}^*\right)rdr, \tag{9}$$

where for example $H_{\phi1}^*$ means complex conjugate to $H_{\phi1}$. Average power propagated inside the core P_1 can be expressed as percentage through the relation $P_{1\%} = [P_1/(P_1 + P_2)] \cdot 100\%$. The expressions for E_z and H_z are given by [3], [5], [6]

$$E_{z1}=A_E J_m\left(\chi r\right)\exp\left[j\left(m\phi+\omega t-\beta z\right)\right], \tag{10}$$

$$H_{z1}=A_H J_m\left(\chi r\right)\exp\left[j\left(m\phi+\omega t-\beta z\right)\right], \tag{11}$$

for the core and [3], [5], [6]

$$E_{z2}=B_E K_m\left(\sigma r\right)\exp\left[j\left(m\phi+\omega t-\beta z\right)\right], \tag{12}$$

$$H_{z2}=B_H K_m\left(\sigma r\right)\exp\left[j\left(m\phi+\omega t-\beta z\right)\right], \tag{13}$$

for the cladding of the step index fiber, where A_E, A_H, B_E and B_H are arbitrary constants, $J_m(\chi r)$ is the Bessel function

T. Kaczmarek is with the Institute of Telecommunications, Photonics and Nanomaterials, Kielce University of Technology, Al. 1000-lecia P.P. 7, 25-314 Kielce, Poland (e-mail: tkaczmar@tu.kielce.pl).

of the first kind of order m and $K_m(\sigma r)$ is the modified Bessel function of the second kind of order m. The constant m must be an integer since the fields must be periodic in ϕ with a period of 2π. Inside the core factor χ^2 is given by [3], [5], [6]

$$\chi^2 = k^2 n_1^2 - \beta^2, \quad (14)$$

while outside the core

$$\sigma^2 = \beta^2 - k^2 n_2^2. \quad (15)$$

Time coordinate T from equation (1), which describes pulse evolution inside a single-mode fiber, is related to t from equations (9), (10), (11) and (12) in the following way [1], [4]

$$T = t - z/v_g = t - \beta_1 z, \quad (16)$$

where v_g is the group velocity at which the frame of reference is moving with the pulse, β_1 is the first derivative of β with respect to ω and is related to group velocity dispersion through well known relation $\beta_2 = \mathrm{d}\beta_1/\mathrm{d}\omega$.

The solution for β from permissible range for guided modes

$$kn_2 \leq \beta \leq kn_1, \quad (17)$$

must be determined from the boundary conditions, which require that the tangential components E_ϕ and E_z of electric field vector $\vec{E}$ inside and outside of the dielectric interface at $r = a$ must be the same and similarly for the tangential components H_ϕ and H_z of magnetic field vector $\vec{H}$. By requiring the continuity of E_z,H_z, E_ϕ, and H_ϕ at $r = a$, one can obtain a set of four homogeneous equations satisfied by A_E, A_H, B_E and B_H. These equations have a nontrivial solution only if the determinant of the coefficient matrix vanishes. After considerable algebraic details, this condition leads to the following eigenvalue equation for $\beta(\mathrm{EV}(\beta) = 0)$ [5], [6]:

$$\left(\frac{J_m^|(u)}{uJ_m(u)} + \frac{K_m^|(w)}{wK_m(w)}\right)\left(\frac{J_m^|(u)}{uJ_m(u)}n_1^2 + n_2^2\frac{K_m^|(w)}{wK_m(w)}\right) - \left(\frac{\beta m}{k}\right)^2\left(\frac{1}{u^2} + \frac{1}{w^2}\right)^2 = 0. \quad (18)$$

II. Method

Step index fiber modeling in order to soliton propagation can be divided into two stages. In the fist stage the optimal value of the normalized frequency V_{opt} is calculated. In this end, eigenvalue equation (18) for step index fiber is solved numerically. The optimal value of the normalized frequency guarantees that the cut off wavelength λ_C for TE_{01} mode is equal to the zero dispersion wavelength λ_{ZD}, furthermore, if $\lambda_C = \lambda_{ZD}$ then also $\Delta\lambda_C = \Delta\lambda_{ZD}$, where $\Delta\lambda_C = \lambda_{opt} - \lambda_C$ and similarly $\Delta\lambda_{ZD} = \lambda_{opt} - \lambda_{ZD}$ ($\lambda_{opt} = 1.55\mu$m is optimal operating wavelength). In this special case, single mode condition $\lambda_{oper} > \lambda_C$ is in full agreement with bright soliton propagation condition $\lambda_{oper} > \lambda_{ZD}$, where λ_{oper} is operating wavelength. If $V > V_{opt}$ then $\lambda_C > \lambda_{ZD}$ which means that $\Delta\lambda_C < \Delta\lambda_{ZD}$ and simultaneous fulfillment of single mode and bright soliton propagation condition is only possible for $\lambda_{oper} > \lambda_C$. Similarly if $V < V_{opt}$, then $\lambda_{ZD} > \lambda_C$ ($\Delta\lambda_{ZD} < \Delta\lambda_C$) and simultaneous fulfillment of single mode working regime and pulse like soliton propagation condition is possible if and only if $\lambda_{oper} > \lambda_{ZD}$ (TABLE III).

TABLE I
SELLMEIER COEFFICIENTS VALUES FOR APPROPRIATE GERMANIUM DIOXIDE MOL % DOPING OF SILICA GLASS AND FOR PURE SILICA GLASS [6], [7]

	100m% SiO_2	3.1m% GeO_2	5.8m% GeO_2	7.9m% GeO_2	13.5m% GeO_2
a_1	0.69616	0.70285	0.70888	0.71368	0.71104
a_2	0.40794	0.41463	0.42068	0.42548	0.45188
a_3	0.89749	0.89745	0.89565	0.89642	0.70404
λ_1 [μm]	0.06840	0.07277	0.06090	0.06171	0.06427
λ_2 [μm]	0.11624	0.11430	0.12545	0.12708	0.12940
λ_3 [μm]	9.89616	9.89616	9.89616	9.89616	9.42547

TABLE II
FOUR CASES OF CORE AND CLADDING CHEMICAL COMPOSITION OF STEP INDEX FIBER

Case	Core	Cladding
1	3.1mol% GeO_2 & 96.9mol% SiO_2	100mol% SiO_2
2	5.8mol% GeO_2 & 94.2mol% SiO_2	100mol% SiO_2
3	7.9mol% GeO_2 & 92.1mol% SiO_2	100mol% SiO_2
4	13.5mol% GeO_2 & 86.5mol% SiO_2	100mol% SiO_2

If one starts from value 2.4 for normalized frequency and tries to calculate the optmal value of core radius of the fiber which cladding is made of pure SiO_2 and its core is doped by different mol % GeO_2, one has to use the following relation [3], [5], [6]

$$a = V/\left(k(\lambda)\sqrt{n_1^2(\lambda) - n_2^2(\lambda)}\right), \quad (19)$$

where $V = 2.4$ is the normalized frequency, $k = 2\pi/\lambda$ is the wave number, n_1 and n_2 are refractive indices of the core and cladding, respectively. The values of both indices are determined through Sellmeier dispersive formula [3], [6], [7]

$$n = \sqrt{1 + \sum_{i=1}^{3}\frac{a_i\lambda^2}{\lambda^2 - \lambda_i^2}}, \quad (20)$$

where a_i is the oscillator strength, λ_i is the oscillator resonance wavelength. Both coefficients values for appropriate GeO_2 mol % doping of SiO_2 are presented in TABLE I.

By the assumption that the cladding is made of pure silica glass there are four cases in the modeling of step index fiber for four types of germanium dioxide doping, which can be numbered in increasing GeO_2 doping order (TABLE II).

After suitable rearranging of equation (19) to the following form $\lambda = 2\pi a\left(n_1^2(\lambda) - n_2^2(\lambda)\right)^{1/2}/V$, it is possible to calculate cut off wavelength λ_C for the TE_{01} mode. Obtaining of zero dispersion wavelength λ_{ZD} can be done in two ways. By the use of group velocity dispersion $\beta_2 = f(\lambda)$ or dispersion parameter $D = f(\lambda)$ characteristic. In each case the result should be the same.

In the second stage, nonlinear Schrödinger equation is solved numerically by the use of split-step Fourier (SSF) method, for each case of the optimized step index fiber

TABLE III
INTERMIDIET AND FINAL RESULTS OF THE FIRST STAGE MODELING PROCESS

Normalized Frequency	Case 1 $\lambda[\mu m]$	Case 2 $\lambda[\mu m]$	Case 3 $\lambda[\mu m]$	Case 4 $\lambda[\mu m]$
$V = 2.4$	λ_C=1.547 λ_{ZD}=1.287	λ_C=1.547 λ_{ZD}=1.295	λ_C=1.547 λ_{ZD}=1.310	λ_C=1.547 λ_{ZD}=1.380
V=2.3	λ_C=1.483 λ_{ZD}=1.291	λ_C=1.483 λ_{ZD}=1.303	λ_C=1.483 λ_{ZD}=1.323	λ_C=1.480 λ_{ZD}=1.408
V_{opt}=2.231				λ_C=λ_{ZD}= =1.434
V=2.2	λ_C=1.420 λ_{ZD}=1.296	λ_C=1.419 λ_{ZD}=1.314	λ_C=1.419 λ_{ZD}=1.340	λ_C=1.414 λ_{ZD}=1.448
V_{opt}=2.107			λ_C=λ_{ZD}= =1.360	
V=2.1	λ_C=1.356 λ_{ZD}=1.302	λ_C=1.354 λ_{ZD}=1.327	λ_C=1.355 λ_{ZD}=1.362	
V_{opt}=2.065		λ_C=λ_{ZD}= =1.333		
V_{opt}=2.024	λ_C=λ_{ZD}= =1.308			
V=2.0	λ_C=1.292 λ_{ZD}=1.310	λ_C=1.291 λ_{ZD}=1.345		

TABLE IV
SUMMARIZED RESULTS FOR THE FIRST STAGE OF STEP INDEX FIBER MODELING PROCESS

Parameter	Case 1	Case 2	Case 3	Case 4
V_{opt}	2.024	2.065	2.107	2.231
a $[\mu m]$	4.293	3.178	2.762	2.200
$P_{1\%}$ [%]	60.96	62.74	64.49	67.26
ω_{eff} $[\mu m]$	5.228	3.815	3.270	2.510
A_{eff} $[\mu m^2]$	86.86	45.73	33.60	19.79
γ $[1/Wkm]$	1.039	1.950	2.654	4.507
$\lambda_C = \lambda_{ZD}$ $[\mu m]$	1.308	1.333	1.360	1.434
$\Delta\lambda_C = \Delta\lambda_{ZD}$ = $[nm]$	242.3	217.2	189.9	115.8
D $[ps/km\ nm]$	16.86	13.34	10.61	5.693
β_2 $[ps^2/km]$	-21.51	-17.02	-13.53	-7.263

TABLE V
FOUR PARAMETERS VALUE CALCULATED FOR FOUR CASES OF STEP INDEX FIBER FOR FUNDAMENTAL SOLITON INITIAL WIDTH $T_0 = 1\,ps$.

Parameter	Case 1	Case 2	Case 3	Case 4
P_0 $[W]$	20.71	8.725	5.098	1.612
A_0	4.551	2.954	2.258	1.269
L_D $[m]$	46.49	58.77	73.89	137.7
z_0 $[m]$	73.03	92.32	116.1	216.3

in the first stage, for soliton pulses up to the sixth order. Split-step Fourier is a pseudospectral method, which has been extensively used to solve the pulse-propagation problem in nonlinear dispersive media. In this method approximate solution is obtained by the assumption that in propagating the optical field over a small distance h, the dispersive and nonlinear effects act independently. It can be understood if Eq. (1) is rewritten in the following form [1], [4]

$$\frac{\partial A}{\partial z} = (\mathbf{D} + \mathbf{N})\, A, \qquad (21)$$

where $\mathbf{D} = -(j\beta_2/2)(\partial^2/\partial T^2)$ is a differential operator that accounts for dispersion in a linear medium and $\mathbf{N} = j\gamma|A|^2$ is a nonlinear operator that governs the effect of fiber nonlinearities on pulse propagation. So in case of SSF method optical field propagation from $z$to $z+h$ is carried out in two steps. In the first step $\mathbf{D} = 0$ in Eq. (21) and nonlinearity acts alone, in the second step $\mathbf{N} = 0$ in Eq. (21) and dispersion acts alone. Mathematically it can be prescribed as follows [1], [4]

$$A(z{+}h, T) \approx \mathbf{F}^{-1}\{\exp\left[h\mathbf{D}(j\omega)\right] \mathbf{F}\left[\exp(h\mathbf{N})A(z, T)\right]\}, \quad (22)$$

where $\mathbf{F}$ denotes the Fourier-transform operation, $\mathbf{D}(j\omega) = j\omega^2\beta_2/2$ is obtained from a differential operator by replacing $\partial/\partial T$ with $j\omega$, where ω is the frequency in the Fourier domain.

III. Results

Searching the optimal value of the normalized frequency V_{opt} was started from $V = 2.4$ and closed for $V = 2.0$ (V $\in \{2.4, 2.3, 2.2, 2.1, 2.0\}$). Intermediate ($\lambda_C \neq \lambda_{ZD}$) and final ($\lambda_C = \lambda_{ZD}$) results are presented in TABLE III.

Summarized results for the first stage of step index fiber modeling process for $\lambda_{opt} = 1.55$ μm and for HE_{11} mode are presented in TABLE IV.

In order to solve Eq. (1) numerically for initial condition of the form [1]–[4] $A(z = 0, T) = A_0$ sech(T/T_0), it is necessary to calculate peak amplitude value A_0 (which is proportional to peak power P_0) for appropriate soliton order N from the following relation [1]–[4] $N^2 = \gamma P_0 L_D$, where $L_D = T_0^2/|\beta_2|$ is the dispersion length and T_0 is the measure of the impulse width. For fundamental ($N = 1$) and higher order solitons ($N = 2, 3, 4, \ldots$), it is possible to calculate soliton period z_0 from the dispersion length value L_D obtained earlier because [1]–[4] $z_0 = (\pi/2)L_D$.

Only fundamental soliton ($N = 1$) can be used as information bits in soliton-based communication systems and only when individual solitons are well isolated (RZ format). The last requirement can be used to relate the soliton width T_0 to the bit rate B as follows [2]–[4] $B = 1/T_B = 1/(2q_0T_B)$, where T_B is the duration of the bit slot and $2q_0 = T_B/T_0$ is the separation between neighboring solitons in normalized units. For $T_0 = 1$ *ps* and $q_0 = 5$, bit rate B in soliton based communication system is equal to $B = 100$ *Gbit/s*. Table V shows calculation results for four necessary parameters needed to solve numerically Eq. (1), for initial width $T_0 = 1$ *ps* and for fundamental soliton ($N = 1$).

IV. Discussion

Fig. 1 shows lack of the shape variation of the pulse as a function of the propagation distance (one soliton period which is equal to $z_0 = 216.3$ m) for the fundamental soliton in case of number 4. It means that first-order soliton ($N = 1$) can be generated for peak amplitude value $A_0 = 1.269$ (column 5 of TABLE V).

V. Conclusions

On the basis of the performed calculations it has been found that if mol % doping of germanium dioxide is increasing inside

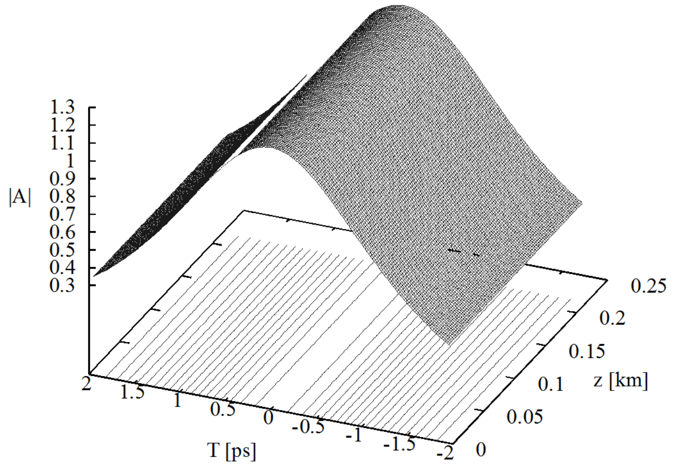

Fig. 1. Evolution of the first-order soliton (N = 1) over one soliton period.

the core, then the optimal value of the normalized frequency V_{opt} of the modeled step index fiber is also increasing. Increase of V_{opt} implies increase of zero dispersion wavelength λ_{ZD} and cut off wavelength λ_C, which are equal in case of normalized frequency optimization. Additionally, growth of V_{opt} value is responsible for rise of the average power curried by the core P_1. There is only one more parameter which value is increasing when mol % doping of germanium dioxide is increasing. It is nonlinear parameter γ, which in turn is responsible for decreasing the peak power needed to generate fundamental soliton in each case of step index fiber modeling process. Furthermore, decrease of dispersion parameter D and absolute value of group velocity dispersion parameter β_2 is responsible for increase of dispersion length L_D and value of the soliton period z_0. Fundamental disadvantage of increasing λ_{ZD} and λ_C is decreasing of bright soliton generation region $\Delta\lambda_{ZD}$ and single mode operation region $\Delta\lambda_C$, which are essential in wavelength division multiplexing technique application.

References

[1] G. P. Agrawal, *Nonlinear Fiber Optics*, third edition ed. Academic Press, 2001.

[2] ——, *Applications of Nonlinear Fiber Optics*. Academic Press, 2001.

[3] ——, *Fiber-Optic Communication Systems*. John Wiley & Sons, 2002.

[4] E. Iannone, F. Matera, A. Mecozzi, and M. Settembre, *Nonlinear Optical Communication Networks*. John Wiley & Sons, 1998.

[5] G. Keiser, *Optical Fiber Communications*. McGraw-Hill, 1991.

[6] A. Majewski, *Teoria i projektowanie Światłowodów*. WNT, Warszawa, 1991, (in Polish).

[7] M. J. Adams, *An Introduction to Optical Waveguides*. John Wiley & Sons, 1981.

Tomasz Kaczmarek received the M.Sc. degree in electrical engineering from Kielce University of Technology in 1994 and the Ph.D. degree in electronic engineering from Warsaw University of Technology in 2002. Currently he is the Head of Laboratory of Optical Fiber Technology of the Institute of Telecommunication, Photonics and Nanomaterials at the Kielce University of Technology. He authored and co-authored over 30 publications. His current research interests include fiber optics and nonlinear fiber optics.