Algorithm for queueing networks with multi-rate traffic

Villy B. Iversen and King-Tim Ko

Abstract—**In this paper we present a new algorithm for evaluating queueing networks with multi-rate traffic. The detailed state space of a node is evaluated by explicit formulæ. We consider reversible nodes with multi-rate traffic and find the state probabilities by taking advantage of local balance. Theory of queueing networks in general, presumes that we have product form between the nodes. Otherwise, we have the state space explosion. Even so, the detailed state space of each node may become very large because there is no product form between chains inside a node. A prerequisite for product form is reversibility which implies that the arrival process and departure process are identical processes, for example state-dependent Poisson processes. This property is equivalent to reversibility. Due to product form, an open network with multi-rate traffic is easy to evaluate by convolution algorithms because the nodes behave as independent nodes. For closed queueing networks with multiple servers in every node and multi-rate services we may apply multi-dimensional convolution algorithm to aggregate the nodes so that we end up with two nodes, the aggregated node and a single node, for which we can calculate the detailed performance measures.**

Index Terms—**multi-rate traffic, queueing networks, reversibility, insensitivity, product form, convolution algorithm**

I. Introduction

IN 1957, J.R. Jackson who was working at production planning and manufacturing systems, published a paper [1] showing that a queueing network of *M/M/n*–nodes has product form. Knowing the fundamental theorem of Burke (1956 [2]) Jackson's result is obvious. Historically, the first paper on queueing systems in series was by another Jackson, R.R.P. Jackson (1954 [3]).

The key point of Jackson's theorem is that each node can be considered to be independent of all other nodes, and that the state probabilities are given by Erlang's waiting time model *M/M/n*. This simplifies the calculation of the state space probabilities significantly. The proof of the theorem was given by Jackson in 1957 by showing that the node balance equations are fulfilled under the assumption of statistical equilibrium. Jackson's first model thus only deals with open queueing networks.

In Jackson's second model (1963 [4]) the arrival intensity from outside may depend on the current number of customers in the network. Furthermore, the service rates may depend on the number of customers k in the nodes. In this way, we can model queueing networks which are either closed, open, or mixed. In all three cases, the state probabilities have product form between nodes. The model by Gordon & Newell from 1967 which is often cited in the literature can be treated as a special case of Jackson's second model.

The theory of queueing networks assumes that a customer samples a new service time in every node. This is a necessary assumption for having product form. This assumption was investigated by Kleinrock (1964 [5]) and it turns out to be a good approximation in real life.

In 1975 the second model of Jackson was further generalized by *B*askett, *C*handy, *M*untz and *P*alacios (1975 [6]) to so-called *BCMP*–networks. These authors showed that queueing networks with K nodes and more than one type of customers also have product form, provided that:

a) The customers are classified into N *chains*. Each chain $j \in N$ is in each node $i \in K$ characterized by its own mean service time s_i^j and routing probabilities $p_{ik}, \{i, k \in K\}$. A customer may change from one chain to another chain with a certain probability after finishing service at a node. If the queueing system of a node is a classical *M/M/n* system (including *M/M/1*), then *the average service time in a node must be identical for all chains.*
b) Each node is a symmetric (= reversible) queueing system mentioned below (Sec. II-B): for each chain a Poisson arrival process implies a Poisson departure process.

BCMP–networks can be evaluated by the multi-dimensional convolution algorithm for multi-server systems. The famous *MVA* (mean value) algorithm by Lavenberg & Reiser [7] is applicable only if all nodes are single server systems or infinite server systems. This paper is based on models of Kingman (1969 [8]) and Sutton (1980 [9]), which are generalizations of Erlang's approach based on the assumption of statistical equilibrium. All derivations are mathematically very simple. Similar models are dealt with by Bonald & Proutière (2003 [10]) and Bonald & Virtamo (2005 [11]). They also consider multi-rate queueing nodes, but only with infinite buffer. They present expressions for the average flow throughput. Serfozo (1999 [12]) presents the general theoretical background.

Our approach is algorithmic and directed to engineering applications. For open networks we achieve an algorithm which is linear in both number of traffic streams and number of channels and has a very small memory requirement (Iversen, 2007 [13]). For loss (buffer-less) systems we obtain an algorithm for BPP (Binomial–Poisson–Pascal) traffic with individual performance measures for each stream. For systems with buffers (finite or infinite) we obtain both mean virtual

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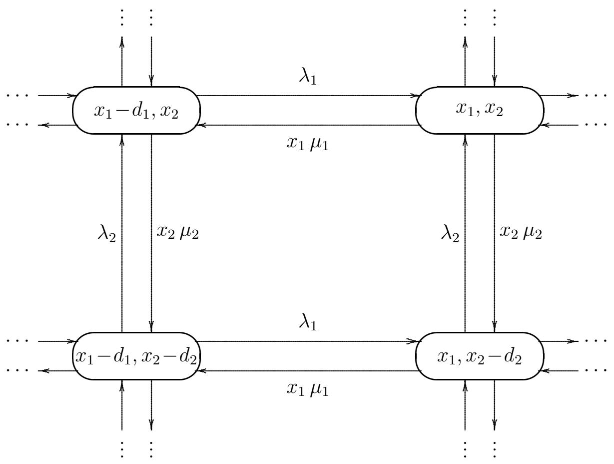

Fig. 1. *State-transition diagram for four neighboring states in a reversible node with two multi-rate traffic streams and infinite capacity. State x_j denoters that x_j channels are occupied by type j calls. For type j with slot-size d_j we choose the service rate $d_j\,\mu_j$.*

waiting times, virtual mean queue lengths, and loss (overflow) probability for each stream.

In this paper we derive explicit formulæ for detailed state probabilities of multi-rate closed queueing networks.

II. Reversible multi-server multi-service nodes

We consider a system with n servers (channels, bandwidth units) and infinite queue. The system is offered N different traffic streams. Customers of type j arrive to the system according to a Poisson arrival process with intensity λ_j. A customer of type j attempts to occupy d_j servers, and if all these are obtained the service time is exponentially distributed with intensity $d_j\,\mu_j$ $(j = 1, 2, \ldots, N)$ where the factor d_j only is chosen for convenience. Later we may choose $\mu_j = \mu$ for all services, then the service rate in a state with a total of x busy channels will be equal to $x\,\mu$, independent of the actual mix of services. The state of the system is defined by $\underline{x} = (x_1, x_2, \ldots, x_j, \ldots, x_N)$, where x_j is the number of channels and/or queueing positions occupied by type j customers. Thus, the number of type j customers is equal to x_j/d_j. If the total demand for channels is bigger than the number of channels available, then the customers share the capacity and queueing positions in some particular way which is specified below. When the number of servers is infinite, we get the state transition diagram shown for two traffic streams in Fig. 1. This diagram is reversible and has a simple product form solution.

However, the capacity is limited to n servers, so we have to reduce the service rates in all states requiring more than n servers (overload). In the following we illustrate the theory for two services, but also mention the general case with N different services.

A. Reduction factors

We consider a system with N traffic streams. Let:

$$\begin{aligned}\underline{x} &= (x_1, x_2, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_N)\\ \underline{x} - d_j &= (x_1, x_2, \ldots, x_{j-1}, x_j - d_j, x_{j+1}, \ldots, x_N)\end{aligned}$$

For type j customers the service rate in state $\underline{x} = (x_1, x_2, \ldots, x_j, \ldots, x_N)$ is reduced by a factor $g_j(\underline{x})$. The reduction factors $g_j(\underline{x})$ are chosen so that we maintain reversibility, and they can be specified for various parts of the state transition diagram as follows, where (d) is the only non-trivial case.

(a) $x_i \leq 0: \quad g_j(\underline{x}) = 0$.
The reduction factors are undefined for these states for which the state probabilities are zero. By choosing the value zero, the recursion formulæ below are correctly initiated.

(b) $x_i \geq 0 \;\wedge\; 0 < \sum_{j=1}^{N} x_i \leq n: \quad g_j(\underline{x}) = 1$.
Every call is allocated the capacity required and there is no reduction.

(c) $x_j = 0$ for all $j \neq i, x_i \geq n: \quad g_i(\underline{x}) = n/x_i$
Along the axes we have a classical *M/M/n*–system with only one service, and the calls share the capacity equally.

(d) States with more types of customers in total requiring more than n channels. If possible, we want to choose $g_i(x_1, x_2)$ so that all capacity is used. This requirement implies:

$$\sum_{j=1}^{N} x_j \cdot g_j(\underline{x}) = n\,, \qquad \sum_{j=1}^{N} x_j \geq n\,. \tag{1}$$

We consider states $\underline{x}$ where $x = \sum_{i=1}^{N} x_i > n$ and all capacity is used. We apply flow balance equations for Kolmogorov cycles.

A necessary and sufficient condition for reversibility (Kingman, 1969 [8], Sutton, 1980 [9]) is that all two-dimensional flow paths are in equilibrium. In total we may choose:

$$\binom{N}{2} = \frac{N\,(N-1)}{2}$$

different cycles and thus different balance equations.

We assume that we know the reduction factors for states $\underline{x} - d_j$ below state $\underline{x}$. To find the N reduction factors in state $\underline{x} = \{x_1, x_2, \ldots, x_N\}$ we need N independent equations. Thus, we may choose Kolmogorov cycles for the two-dimensional planes $\{1, j\}, (j = 2, 3, \ldots, N)$ which yields $N-1$ independent equations. Furthermore we have the normalization equation (1) requiring that the total capacity used during overload is n. We get the following flow balance equations for $j = 2, 3, \ldots N$:

$$g_1(\underline{x}) \cdot g_j(\underline{x} - d_1) = g_j(\underline{x}) \cdot g_1(\underline{x} - d_j)$$

or

$$g_j(\underline{x}) = g_1(\underline{x}) \cdot \frac{g_j(\underline{x} - d_1)}{g_1(\underline{x} - d_j)}\,, \quad j = 2, 3, \ldots, N\,. \tag{2}$$

The capacity normalization equations is (1):

$$
\begin{aligned}
n &= \sum_{i=1}^{N} x_i \cdot g_i(\underline{x}) \\
&= \sum_{i=1}^{N} \left\{ x_i \cdot g_1(\underline{x}) \cdot \frac{g_i(\underline{x}-d_1)}{g_1(\underline{x}-d_i)} \right\}, \\
g_1(\underline{x}) &= \frac{n}{\sum_{i=1}^{N} \left\{ x_i \cdot \frac{g_i(\underline{x}-d_1)}{g_1(\underline{x}-d_i)} \right\}} . \qquad (3)
\end{aligned}
$$

Thus, we find $g_1(\underline{x})$ from (3) and all other reduction factors in state $\underline{x}$ from (2). As we know, all reduction factors for global states x up to n where $x = \sum_{i=1}^{N} x_i$ and all reduction factors for states where only one service is active, then we can calculate all reduction factors recursively. This is equivalent to calculating the relative state probabilities, and thus by global normalization the normalized state probabilities.

For two traffic streams we get the reduction factors:

$$
\begin{aligned}
g_1(x_1,x_2) &= \frac{n}{x_1 + x_2 \cdot \frac{g_2(x_1-d_1,x_2)}{g_1(x_1,x_2-d_2)}} \\
g_2(x_1,x_2) &= \frac{n}{x_2 + x_1 \cdot \frac{g_1(x_1,x_2-d_2)}{g_2(x_1-d_1,x_2)}} .
\end{aligned}
$$

We always find a unique solution when the offered traffic is less than the capacity. When we know the reduction factors, it is easy to find the relative state probabilities by local balance equations. Due to reversibility we have local balance for each service. We get

$$\lambda_j \cdot p(\underline{x}-d_j) = g_j(\underline{x})\, x_j\, \mu_j \cdot p(\underline{x}), \quad j = 1,\ldots,N.$$

Thus, we can recursively find all reduction factors and all state probabilities expressed by state zero. By normalization, which should be done in each step to ensure numerical accuracy, we obtain the absolute state probabilities for a single multi-rate n-server node. From the state probabilities we find performance measures as mean sojourn time and throughput.

For low and normal load, each connection will almost get the required bandwidth as for proportional fair scheduling. It can be shown by studying the reduction factors for increasing load that if the system becomes highly overloaded, then in the limit we get fair scheduling where all connections will be allocated the same capacity independent of the required slot-sizes.

B. Classical queueing networks as special cases

When all traffic streams have the same bandwidth demand $d_j = 1$, we get the following simple solution:

$$g_j(\underline{x}) = \frac{n}{\sum_{j=1}^{N} x_j}, \quad \sum_{j=1}^{N} x_j \geq n, \quad j = 1,2,\ldots,N. \qquad (4)$$

Thus, the service rates of all customers are reduced by the same factor and during overload the customers share the capacity equally. The state transition diagram can be interpreted as modeling the following systems, illustrated by Fig. 2.

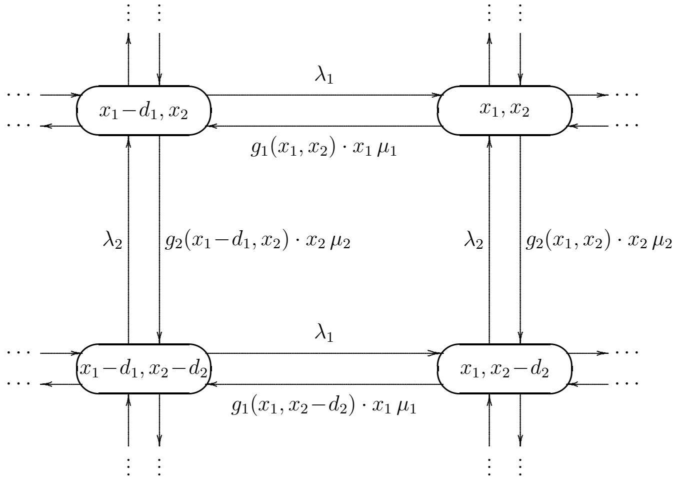

Fig. 2. *State-transition diagram for four neighboring states in a reversible system with two multi-rate traffic streams.*

C. Generalized processor sharing (GPS) system

In states (x_1,x_2) below saturation $(x_1+x_2 \leq n)$ every user occupy one server. Above saturation all users share the available capacity equally. The model is insensitive to the service time distribution and each service may have individual mean service time. This model is called the *GPS* = Generalized Processor Sharing model. For states $x_1+x_2 > n$, a customer type one wants a total service rate μ_1, and a customer type two wants a service rate μ_2. But the service rates of all customers are reduced by the same factor $n/(x_1+x_2)$.

As a special case, the Σ_j *M_j/G_j/1–PS* single-server system with processor sharing (PS) is reversible and insensitive to the service time distributions and each class may have individual mean service times.

D. Classical multi-service multi-server system

In classical queueing, a customer always get one server for exclusive usage. Then to maintain reversibility for $x_1+x_2 > n$ we have to require that all customers have the same service rate $\mu_j = \mu$, and thus the same exponentially distributed service time. The mathematical proof will be give elsewhere. This corresponds to an *M/M/n* system with total arrival rate $\lambda = \sum_j \lambda_j$ and service rate μ. All customers in the system have the same probability of being the next one departing, independent of the call type.

The system *M/G/∞* is reversible, as the departure process is a Poisson process because it is a random translation of the arrival process.

E. $\Sigma_j M_j/G_j$/1–LCFS–PR single-server system

From the very nature of the model it is obvious that it is reversible as we always follow the same path back to state zero as away from state zero. It is insensitive to the service time distribution and each service may have individual mean service time.

In conclusion, multi-server queueing systems with more classes of customers, all having bandwidth demand $d_j =$

$1, (j = 1, 2, \ldots, N)$, will only be reversible when the system is one of the following queueing system:

- *M/G/n–GPS* , (for $n = 1$ PS - Processor Sharing).
- *M/G/1–LCFS–PR*,
- *M/M/n*, including *M/M/1* with same service time for all customers.

These systems are also called symmetric queueing systems. Reversibility implies that the departure processes are Poisson processes for all classes. These systems make up the nodes allowed in *BCMP*–queueing networks.

III. Multi-rate multi-server queueing networks

We now build a network of nodes of the above type. As in ordinary queueing networks we have a routing matrix which specifies the route followed by a given traffic stream, named a chain.

Open Queueing Networks

These are easy to deal with. By solving the flow balance equations (flow in = flow out for each node) for each chain we find the load (offered traffic) of each node. We then find the state probabilities of each node and due to product form we easily get the state probabilities of the total queueing network.

Closed Queueing Networks

From the flow balance equations we first find the relative load of each chain in each node. For a closed network we aggregate the nodes by multi-dimensional convolutions, keeping account of the number of customers in each chain in the aggregated node. All nodes except the target node are aggregated into one node. This aggregated node is convolved with the target node for which we find the performance measures during the convolution. The multi-dimensional convolution is defined as follows:

$$p_{1,2}(x_1, x_2, \ldots, x_N) = p_1 * p_2 = \sum_{i_1=0}^{x_1} \sum_{i_2=0}^{x_2} \cdots \sum_{i_N=0}^{x_N} p_1(x_1-i_1, x_2-i_2, \ldots, x_N-i_N) \cdot p_2(i_1, i_2, \ldots, i_N)$$

The parameter x_j is given in number of channels. Both x_j and i_j belong to $\{0, d_j, 2d_j, \ldots, d_j\, S_j\}$, where S_j is the number of customers in chain j, and $j = 1, 2, \ldots, N$. By changing the order of convolution we obtain the performance measures for each node. The performance measures are for example mean waiting time and mean queue length for each chain, and carried traffic for each stream in the node.

IV. Numerical example

The convolution algorithm for closed multi-rate queueing networks has been inplemented in a master thesis project by Iliakis & Kardaras (2007 [14]).

Let us consider a closed network with two nodes and two types of customers, alternating between the two nodes (a generalized machine-repair model). We have 10 type-1 customers each requesting one channel for service in a node. The service time is 4 tu in node-1 and 1 tu in node-2 (tu = time units). We have 5 type-2 customers each requesting two channels for service in a node. The service time is 2 tu in node-1 and 0.5 tu in node-2. Thus, the packet size (bandwidth times mean service time) is the same for the two types.

The capacity is 20 channels in node-1 so that we never experience delay. The sojourn time in node-1 is thus 4 tu, respectively 2 tu. In node-2 we have 5 channels and infinite queue. We find the sojourn times equal to 1.1929 tu, respectively 0.6990 tu. Thus, the virtual mean waiting time (increase in sojourn time due to limited capacity) in node-2 is 0.1929 tu for type-1, and 0.1990 for type-2. The waiting times of the two services are of same order of size, but by allocating bigger bandwidth to a type of traffic we can reduce the sojourn time (response time).

Limitations of the algorithm are the number of states in the multi-dimensional state space of each node, and the number of operations required during the multi-dimensional convolution of the nodes.

V. Future work

The above model corresponds to a store-and-forward packet switched network with bottlenecks and to models considered in production systems. The model may be generalized in several ways. We way reserve at fixed minimum bandwidth to a certain type in each node this type visits, so that we have an end-to-end dedicated path with a guaranteed bandwidth as in ATM and MPLS networks. If the stream requires more bandwidth than guaranteed, then it has to compete with other streams in each node for additional bandwidth. The system will still be reversible.

We may introduce an upper limit to the number of simultaneous connections of each type in a node. Then we may experience blocking in the nodes, and the system will only be reversible if we include blocked calls in the departure process.

VI. Conclusions

The convolution algorithm for the closed multi-rate queueing networks is a generalization of the classical convolution algorithm for queueing networks and has similar limitations in number of chains and customers in each chain.

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